J. Math. Soc. Japan
Vol. 73, No. 2 (2021) pp. 403–431
doi: 10.2969/jmsj/79867986

BHK mirror symmetry for K3 surfaces with non-symplectic automorphism

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(Received Feb. 14, 2018)
(Revised Oct. 29, 2019)

Abstract. In this paper we consider the class of K3 surfaces defined as hypersurfaces in weighted projective space, that admit a non-symplectic automorphism of non-prime order, excluding the orders 4, 8, and 12. We show that on these surfaces the Berglund–Hübsch–Krawitz mirror construction and mirror symmetry for lattice polarized K3 surfaces constructed by Dolgachev agree; that is, both versions of mirror symmetry define the same mirror K3 surface.

Introduction.

Since its discovery by physicists nearly 30 years ago, mirror symmetry has been the focus of much interest for both physicists and mathematicians. Although mirror symmetry has been "proven" physically, we have much to learn about the phenomenon mathematically. When we speak of mirror symmetry mathematically, there are many different constructions or rules for determining when a Calabi–Yau manifold is "mirror" to another. The constructions are often formulated in terms of families of Calabi–Yau manifolds. A natural question is whether, in a situation where more than one version can apply, they produce the same mirror (or mirror family). In this article, we consider two versions of mirror symmetry for K3 surfaces, and show that in this case the answer is affirmative, as we might expect.

The first version of mirror symmetry of interest to us is known as BHK mirror symmetry. This was formulated by Berglund–Hübsch [**10**], Berglund–Henningson [**9**] and Krawitz [**23**] for Landau–Ginzburg models. Using the ideas of the Landau–Ginzburg/Calabi–Yau correspondence, BHK mirror symmetry also produces a version of mirror symmetry for certain Calabi–Yau manifolds (see Section 2).

In the BHK construction, one starts with a quasihomogeneous and invertible polynomial W and a group G of symmetries of W satisfying certain conditions (see Section 2.2 for more details). From this data, we obtain the Calabi–Yau (orbifold) defined as the hypersurface $Y_{W,G} = \{W = 0\}/G$. Given an LG pair (W, G), BHK mirror symmetry allows to obtain another LG pair (W^T, G^T) satisfying the same conditions, and therefore another Calabi–Yau (orbifold) Y_{W^T,G^T}. We say that $Y_{W,G}$ and Y_{W^T,G^T} form a BHK mirror pair. In our case, we resolve singularities to obtain K3 surfaces $X_{W,G}$ and X_{W^T,G^T},

2010 *Mathematics Subject Classification.* Primary 14J28; Secondary 14J33, 14J17, 11E12, 14J32.
Key Words and Phrases. K3 surfaces, mirror symmetry, mirror lattices, Berglund–Hübsch–Krawitz construction.
The first author has been partially supported by Proyecto Fondecyt Postdoctorado N. 3150015, Proyecto Fondecyt Iniciación en Investigación N. 11190428 and Proyecto Anillo ACT 1415 PIA Conicyt.

which we call a BHK mirror pair. When no confusion arises, we will denote these mirror K3 surfaces simply by X and X^T, respectively.

Another form of mirror symmetry for K3 surfaces, which we will call LPK3 mirror symmetry, is described by Dolgachev in [**17**]. LPK3 mirror symmetry says that the mirror family of a given K3 surface admitting a polarization by a lattice M is the family of K3 surfaces polarized by the *mirror lattice* $M^\vee$. We say that the two K3 surfaces are LPK3 mirror when they are lattice polarized and they belong to LPK3 mirror families (see details in Section 2.1).

Returning to the question posed earlier, one can ask whether the BHK mirror symmetry and LPK3 mirror symmetry produce the same mirror. A similar question was considered by Belcastro in [**8**]. She considers a family of K3 surfaces that arise as (the resolution of) hypersurfaces in weighted projective space, uses the Picard lattice of a general member of the family as polarization, and finds that this particular polarization does not yield very many mirror families.

This polarization fails to yield mirror symmetry for at least two reasons. First, it does not consider the group of symmetries. And secondly—and perhaps more compelling—a result proved by Lyons–Olcken (see [**25**]) following Kelly (see [**22**]) shows that the rank of the Picard lattice of $X_{W,G}$ does not depend on G at all. This fact suggests that we need a finer invariant than the full Picard lattice to exhibit LPK3 mirror symmetry. We need to find a polarizing lattice that recognizes the role of the group G.

The correct polarizing lattice seems to be the invariant lattice

$$S_X(\sigma) = \{x \in H^2(X, \mathbb{Z}) : \sigma^* x = x\}$$

of a certain non-symplectic automorphism $\sigma \in \operatorname{Aut} X$. This was proven in [**3**] and [**14**] in the case of K3 surfaces admitting a non-symplectic automorphism of prime order.

In what follows, we generalize the results of [**3**] and [**14**] to K3 surfaces admitting a non-symplectic automorphism σ of any finite order, excepting orders 4, 8 and 12. By polarizing each of the K3 surfaces in question by the invariant lattice $S_X(\sigma)$ of a non-symplectic automorphism σ of finite order, we prove that BHK mirror symmetry and LPK3 mirror symmetry agree. This is done as in the previous works, by showing that $S_{X^T}(\sigma^T)$ is the mirror lattice of $S_X(\sigma)$.

This situation differs significantly from the case of prime order automorphism in that the invariant lattice is no longer p-elementary and there is no longer a (known) relationship between the invariants of $S_X(\sigma)$ and the fixed locus of σ. Hence, instead of studying the fixed locus in order to recover $S_X(\sigma)$, we determine $S_X(\sigma)$ with other methods. As for orders 4, 8 and 12, more details are required and the methods are slightly different, so that this will be the object of further work.

The question of whether two versions of mirror symmetry produce the same mirror has been investigated by others as well, but for different constructions of mirror symmetry than we consider here. Partial answers to the question are given by Artebani–Comparin–Guilbot in [**4**], where Batyrev and BHK mirror constructions are both seen as specializations of a more general construction based on the definition of good pairs of polytopes. Rohsiepe also considered Batyrev mirror symmetry in connection with LPK3 mirror symmetry in [**29**], where he shows a duality for the K3's obtained as hypersurfaces

in one of the Fano toric varieties constructed by one of the 4319 3-dimensional reflexive polytopes. As in Belcastro's paper [**8**], Rohsiepe used the Picard lattice of a general member of the family of such hypersurfaces to polarize the K3 surfaces. As it turns out, only 14 of the 95 weight systems yield a K3 surface in a Fano ambient space. We do not consider such a restriction in the current paper.

Clarke has also described a framework which he calls an auxiliary Landau–Ginzburg model, which encapsulates several versions of mirror symmetry, including Batyrev–Borisov, BHK, Givental's mirror theorem and Hori–Vafa mirror symmetry (see [**13**]). Kelly also has some results in this direction in [**22**], where he shows by means of Shioda maps, that certain BHK mirrors are birational. The current article is similar in scope to these articles.

There are also several papers treating non-symplectic automorphisms of K3 surfaces, which are closely related to this paper. These include [**5**] for automorphisms of order four, [**15**] for order six, [**31**] for order 2^p, [**1**] for order eight, and [**2**] for order sixteen. In general, it seems difficult to find the invariant lattice of a non-symplectic automorphism on a K3 surface. The current article gives some new methods for computing the invariant lattice, which we hope will yield more general results.

As complementary results, in doing this classification we discovered the existence of one of the cases that couldn't be discovered in the order 16 classification in [**2**] namely a K3 surface admitting a purely non-symplectic automorphism of order sixteen, which has as fixed locus a curve of genus zero, and 10 isolated fixed points. This is number 58 in Table 8. Dillies has also found such an example in [**16**].

Additionally, our computations unearthed a different result from Dillies in [**15**]. If we look at Table 13, we find the invariant lattice for number 29 and one of rows of 5d has an invariant lattice of order 12. These K3 surfaces admit an automorphism of order three, namely σ_6^2, with invariants $(g, n, k) = (0, 8, 5)$, but the automorphism σ_6 fixes one rational curve and 8 isolated points. This is missing from Table 1 in [**15**]. Furthermore, the same can be said for the K3 surfaces in same table which have $v \oplus 4w_{2,1}^1$ as the invariant lattice, namely one of 8b, 8d, 33a, and 33b. These K3 surfaces admit a non-symplectic automorphism of order three with invariants $(g, n, k) = (0, 7, 4)$, but σ_6 fixes one rational curve and seven isolated points. This is also missing from the Table in [**15**].

The paper is organized as follows. In Section 1 we recall some definitions and results on K3 surfaces and lattices, while Section 2 is dedicated to the introduction of mirror symmetry, both LPK3 and BHK. The main result of the paper is Theorem 2.3. Section 3 is dedicated to the explanation of the methods used in the proof. In Section 4 we report some meaningful examples, and Section 5 contains the tables proving the main theorem.

Acknowledgments. The authors would like to thank Michela Artebani, Alice Garbagnati, Alessandra Sarti and Matthias Schütt for many useful discussions and helpful insights. They would also thank Antonio Laface for the help on magma code [**11**].

1. Background.

In this section we recall some facts about K3 surfaces and lattices. For notations and theorems, we follow [**7**], [**27**].

1.1. K3 surfaces.

A *K3 surface* is a compact complex surface X with trivial canonical bundle and $\dim H^1(X, \mathcal{O}_X) = 0$. All K3 surfaces considered here will be projective and minimal.

It is well-known that all K3 surfaces are diffeomorphic and Kähler. Given a K3 surface X, $H^2(X, \mathbb{Z})$ is free of rank 22, the Hodge numbers of X are $h^{2,0}(X) = h^{0,2}(X) = 1$, $h^{1,1}(X) = 20$ and $h^{1,0}(X) = h^{0,1}(X) = 0$, and the Euler characteristic is 24. The Picard group of X coincides with the Néron–Severi group, and both are torsion free.

From the facts above, we see that $H^{2,0}(X)$ is one-dimensional. In fact, it is generated by a nowhere-vanishing two-form ω_X, which satisfies $\langle \omega_X, \omega_X \rangle = 0$ and $\langle \omega_X, \overline{\omega}_X \rangle > 0$.

Given an automorphism σ of the K3 surface X, we get an induced Hodge isometry σ^*, which preserves $H^{2,0}(X)$, i.e., $\sigma^* \omega_X = \lambda_\sigma \omega_X$ for some $\lambda_\sigma \in \mathbb{C}^*$. We call σ *symplectic* if $\lambda_\sigma = 1$ and *non-symplectic* otherwise. If σ is an automorphism with nonprime order m, we say σ is *purely non-symplectic* if $\lambda_\sigma = \xi_m$ with ξ_m a primitive m-th root of unity.

1.2. Lattice theory.

A *lattice* is a free abelian group L of finite rank together with a non-degenerate symmetric bilinear form $B\colon L \times L \to \mathbb{Z}$. A lattice L is *even* if $B(x, x) \in 2\mathbb{Z}$ for each $x \in L$. The *signature* of L is the signature (t_+, t_-) of B. A lattice L is *hyperbolic* if its signature is $(1, \operatorname{rank}(L) - 1)$. A sublattice $L \subset L'$ is called *primitive* if L'/L is free. On the other hand, a lattice L' is an *overlattice* of finite index of L if $L \subset L'$ and L'/L is a finite abelian group. We will refer to it simply as an *overlattice*.

Given a finite abelian group A, a *finite quadratic form* is a map $q : A \to \mathbb{Q}/2\mathbb{Z}$ such that for all $n \in \mathbb{Z}$ and $a, a' \in A$, we have $q(na) = n^2 q(a)$ and $q(a + a') - q(a) - q(a') \equiv 2b(a, a') \pmod{2\mathbb{Z}}$ where $b : A \times A \to \mathbb{Q}/\mathbb{Z}$ is a finite symmetric bilinear form. We define orthogonality on subgroups of A via b.

Given a lattice L, the corresponding bilinear form B induces an embedding $L \hookrightarrow L^*$, where $L^* := \operatorname{Hom}(L, \mathbb{Z})$. The *discriminant group* $A_L := L^*/L$ is a finite abelian group. In fact, if we write B as a symmetric matrix in terms of a minimal set of generators of L, then the order of A_L is equal to $|\det(B)|$. The bilinear form B can be extended to $L^* \times L^*$ taking values in $\mathbb{Q}$. If L is even, this induces a finite quadratic form $q_L : A_L \to \mathbb{Q}/2\mathbb{Z}$.

The minimal number of generators of A_L is called the *length* of L. If A_L is trivial, L is called *unimodular*. For a prime number p, L is called *p-elementary* if $A_L \simeq (\mathbb{Z}/p\mathbb{Z})^a$ for some $a \in \mathbb{N}_0$; in this case, a is the length of A_L.

Two lattices L and K are said to be *orthogonal*, if there exists an even unimodular lattice S such that there is a primitive embedding $L \subset S$ and $L_S^\perp \cong K$. Orthogonality will be a key ingredient in the definition of mirror symmetry for K3 surfaces. The following fact will also be useful.

PROPOSITION 1.1 (cf. [**27**, Corollary 1.6.2]). *Two even lattices L and K are orthogonal if and only if $q_L \cong -q_K$.*

We recall the definition of several lattices that we will encounter later. The lattice U is the hyperbolic lattice of rank 2 whose bilinear form is given by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

The lattices $A_n, D_m, E_6, E_7, E_8, n \geq 1, m \geq 4$ are the even negative definite lattices associated to the respective Dynkin diagrams. For $n \geq 1$, the lattice A_n has rank n and its discriminant group is $\mathbb{Z}/(n+1)\mathbb{Z}$. If p is prime, A_{p-1} is p-elementary (with $a = 1$).

For $m \geq 4$, the lattice D_m has rank m and its discriminant group is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ for m even, and $\mathbb{Z}/4\mathbb{Z}$ for m odd. Finally, E_6, E_7, E_8 have ranks 6, 7, and 8 and discriminant groups of order 3, 2, and 1, respectively.

For $p \equiv 1 \pmod 4$ the lattice H_p is the hyperbolic even lattice of rank 2, whose bilinear form is given by the matrix

$$H_p = \begin{pmatrix} (p-1)/2 & 1 \\ 1 & -2 \end{pmatrix}.$$

The discriminant group of H_p is $\mathbb{Z}/p\mathbb{Z}$.

There are two non–isomorphic hyperbolic lattices of rank 2 with discriminant group $\mathbb{Z}/9\mathbb{Z}$ defined by the matrices

$$L_9 = \begin{pmatrix} -2 & 1 \\ 1 & 4 \end{pmatrix}, \qquad M_9 = \begin{pmatrix} -4 & 5 \\ 5 & -4 \end{pmatrix}.$$

Following [**8**] we recall that $T_{p,q,r}$ with $p, q, r \in \mathbb{Z}$ is the lattice determined by a graph which has the form of a T, and p, q, r are the respective lengths of the three legs. The rank of $T_{p,q,r}$ is $p+q+r-2$ and the discriminant group has order $pqr-pq-qr-pr$.

Given a lattice L, we denote by $L(n)$, the lattice with the same rank as L, but whose values under the bilinear form B are multiplied by n.

Many even lattices are uniquely determined by their rank and the discriminant quadratic form. To make this statement precise, we introduce the following finite quadratic forms. The notation follows [**8**] and the results are proven in [**27**].

We define three classes of finite quadratic forms, $w^{\epsilon}_{p,k}$, u_k, v_k as follows:

1. For $p \neq 2$ prime, $k \geq 1$ an integer, and $\epsilon \in \{\pm 1\}$, let a be the smallest even integer that has ϵ as quadratic residue modulo p. Then we define $w^{\epsilon}_{p,k} : \mathbb{Z}/p^k\mathbb{Z} \to \mathbb{Q}/2\mathbb{Z}$ via $w^{\epsilon}_{p,k}(1) = ap^{-k}$.

2. For $p = 2$, $k \geq 1$ and $\epsilon \in \{\pm 1, \pm 5\}$, we define $w^{\epsilon}_{2,k} : \mathbb{Z}/2^k\mathbb{Z} \to \mathbb{Q}/2\mathbb{Z}$ on the generator via $w^{\epsilon}_{2,k}(1) = \epsilon \cdot 2^{-k}$.

3. For $k \geq 1$ an integer, we define the forms u_k and v_k on $\mathbb{Z}/2^k\mathbb{Z} \times \mathbb{Z}/2^k\mathbb{Z}$ via the matrices:

$$u_k = \begin{pmatrix} 0 & 2^{-k} \\ 2^{-k} & 0 \end{pmatrix}, \qquad v_k = 2^{-k} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

For example, if we consider the lattice $L = A_2$, then $A_L \cong \mathbb{Z}/3\mathbb{Z}$ and q_L has value $4/3$ on the generator. Thus $q_L \cong w^1_{3,1}$.

THEOREM 1.2 (cf. [**27**, Theorem 1.8.1]). *The forms $w^{\epsilon}_{p,k}$, u_k, v_k generate the semigroup of finite quadratic forms.*

In other words every finite quadratic form can be written (not uniquely) as a direct sum of the generators $w^{\epsilon}_{p,k}$, u_k, v_k. Relations can be found in [**27**, Theorem 1.8.2].

For a finite quadratic form q, and a prime number p, we denote q restricted to the p-component $(A_q)_p$ of A by q_p. The following results describe the close link between discriminant quadratic forms and even lattices.

THEOREM 1.3 (cf. [**27**, Theorem 1.13.2]). *An even lattice S with invariants (t_+, t_-, q) is unique if, simultaneously,*

1. $t_+ \geq 1, t_- \geq 1, t_+ + t_- \geq 3$;
2. *for each $p \neq 2$, either* $\operatorname{rank} S \geq 2 + l((A_q)_p)$ *or* $q_p \cong w_{p,k}^{\epsilon} \oplus w_{p,k}^{\epsilon'} \oplus q'_p$;
3. *for $p = 2$, either* $\operatorname{rank} S \geq 2 + l((A_q)_2)$ *or one of the following holds*

$$q_2 \cong u_k \oplus q'_2, \quad q_2 \cong v_k \oplus q'_2, \quad q_2 \cong w_{2,k}^{\epsilon} \oplus w_{2,k}^{\epsilon'} \oplus q'_2.$$

COROLLARY 1.4 (cf. [**27**, Corollary 1.13.3]). *An even lattice S with invariants (t_+, t_-, q) exists and is unique if $t_+ - t_- \equiv \operatorname{sign} q \pmod 8$, $t_+ + t_- \geq 2 + l(A_q)$, and $t_+, t_- \geq 1$.*

COROLLARY 1.5 (cf. [**27**, Corollary 1.13.4]). *Let S be an even lattice of signature (t_+, t_-). If $t_+ \geq 1$, $t_- \geq 1$ and $t_+ + t_- \geq 3 + l(A_S)$, then $S \cong U \oplus T$ for some lattice T.*

In Table 1, we list the discriminant form associated to each of the lattices appearing in our calculations (see Sections 4, 5). A complete description can be found in [**8**, Appendix A].

Table 1. Lattices and forms.

L	$\operatorname{sign} L$	q_L	L	$\operatorname{sign} L$	q_L	L	$\operatorname{sign} L$	q_L
U	(1,1)	trivial	D_6	(0,6)	$(w_{2,1}^1)^2$	$T_{4,4,4}$	(1,9)	v_2
$U(2)$	(1,1)	u	D_9	(0,9)	$w_{2,2}^{-1}$	$T_{3,4,4}$	(1,8)	$w_{2,3}^5$
A_1	(0,1)	$w_{2,1}^{-1}$	E_6	(0,6)	$w_{3,1}^{-1}$	$T_{2,5,6}$	(1,10)	$w_{2,3}^{-5}$
A_2	(0,2)	$w_{3,1}^1$	E_7	(0,7)	$w_{2,1}^1$	$\langle 2 \rangle$	(1,0)	$w_{2,1}^1$
A_3	(0,3)	$w_{2,2}^5$	E_8	(0,8)	trivial	$\langle 4 \rangle$	(1,0)	$w_{2,2}^1$
$A_1(2)$	(0,1)	$w_{2,2}^{-1}$	H_5	(1,1)	$w_{5,1}^{-1}$	$\langle 8 \rangle$	(1,0)	$w_{2,3}^1$
D_4	(0,4)	v	L_9	(1,1)	$w_{3,2}^1$	$\langle -8 \rangle$	(0,1)	$w_{2,3}^{-1}$
D_5	(0,5)	$w_{2,2}^{-5}$	M_9	(1,1)	$w_{3,2}^{-1}$			

Let L' be an overlattice of the lattice L. We call $H_{L'} := L'/L$. By the chain of embeddings $L \subset L' \subset (L')^* \subset L^*$ one has $H_{L'} \subset A_L$ and $A_{L'} = ((L')^*/L)/H_{L'}$.

PROPOSITION 1.6 (cf. [**27**, Proposition 1.4.1]). *The correspondence $L' \leftrightarrow H_{L'}$ is a 1:1 correspondence between even overlattices of finite index of L and q_L-isotopic subgroups of A_L, i.e., subgroups on which the form q_L is 0. Moreover, $H_{L'}^{\perp} = (L')^*/L$ and $q_{L'} = (q_{L|H_{L'}^{\perp}})/H_{L'}$.*

1.3. K3 lattices.

Let X be a K3 surface. It is well-known that $H^2(X,\mathbb{Z})$ is an even unimodular lattice of signature $(3,19)$. As such, it is isometric to the *K3-lattice* $L_{\mathrm{K3}} = U^3 \oplus (E_8)^2$.

We let

$$S_X = H^2(X,\mathbb{Z}) \cap H^{1,1}(X,\mathbb{C})$$

denote the *Picard* lattice of X in $H^2(X,\mathbb{Z})$ and $T_X = S_X^{\perp}$ denote the *transcendental lattice*.

Let σ be a non-symplectic automorphism of X. We let $S_X(\sigma) \subseteq H^2(X,\mathbb{Z})$ denote the σ^*-invariant sublattice of $H^2(X,\mathbb{Z})$:

$$S_X(\sigma) = \{x \in H^2(X,\mathbb{Z}) : \sigma^* x = x\}.$$

One can check that it is a primitive sublattice of $H^2(X,\mathbb{Z})$. In fact, $S_X(\sigma)$ is a primitive sublattice of S_X and in general $S_X(\sigma) \subsetneq S_X$. We let $T_X(\sigma) = S_X(\sigma)^{\perp}$ denote its orthogonal complement in $H^2(X,\mathbb{Z})$. The signature of $S_X(\sigma)$ is $(1,t)$ for some $t \leq 19$, i.e., $S_X(\sigma)$ is hyperbolic.

2. Mirror symmetry.

2.1. Mirror symmetry for K3 surfaces.

Mirror symmetry for a Calabi–Yau manifold X and its mirror $X^\vee$ can be thought of as an exchanging of the Kähler structure on X for the complex structure of $X^\vee$. Thus, a first prediction of mirror symmetry is the rotation of the Hodge diamond:

$$H^{p,q}(X,\mathbb{C}) \cong H^{q,N-p}(X,\mathbb{C})$$

where N is the dimension of X.

For K3 surfaces, however, the Hodge diamond is symmetric under the rotation mentioned above. So we need to consider a refinement of this idea. This is accomplished by the notion of lattice polarization. Roughly, we choose a primitive lattice $M \hookrightarrow S_X$, which plays the role of the Kähler deformations, and the mirror lattice $M^\vee$, which we now define, plays the role of the complex deformations. We will refer to this formulation of mirror symmetry simply as LPK3 mirror symmetry.

Following [**17**], let X be a K3 surface and suppose that M is a lattice of signature $(1,t)$. If $j\colon M \hookrightarrow S_X$ is a primitive embedding into the Picard lattice of X, the pair (X,j) is called an *M-polarized K3 surface*. There is a moduli space of M-polarized K3 surfaces with dimension $19-t$.

We will not be concerned about the embedding. As in [**14**], we will call the pair (X,M) an *M-polarizable* K3 surface if such an embedding j exists. Note that for an M-polarizable K3 surface (X,M), the lattice M naturally embeds primitively into L_{K3}.

Definition 2.1. Let M be a primitive sublattice of L_{K3} of signature $(1,t)$ with $t \leq 18$ such that $M^{\perp}_{L_{\mathrm{K3}}} \cong U \oplus M^\vee$. We define $M^\vee$ to be (up to isometry) the *mirror lattice* of M.[1]

[1] As in [**14**], our definition in this restricted setting is slightly coarser than the one used by Dolgachev in [**17**], since we do not keep track of the embedding $U \hookrightarrow M^{\perp}$ and instead only consider $M^\vee$ up to isometry.

By Theorem 1.3 this definition is independent of the embedding M into L_{K3}. Furthermore under some conditions, (see e.g., Corollary 1.5 and Theorem 1.3) this definition is also independent of the embedding U into $M^{\perp}$. One can check that these conditions are satisfied for the lattices we consider here.

Note that $M^{\vee}$ also embeds primitively into L_{K3} and has signature $(1, 18-t)$. Furthermore, $q_M \cong -q_{M^{\vee}}$. One easily checks that $(M^{\vee})^{\perp}_{L_{\mathrm{K3}}} \cong U \oplus M$.

Given (X, M) an M-polarizable K3 surface and (X', M') an M'-polarizable K3 surface, with M and M' primitive sublattices of S_X and $S_{X'}$, resp., we say that (X, M) and (X', M') are *LPK*3 *mirrors* if $M' = M^{\vee}$ (or equivalently $M = (M')^{\vee}$).

Notice that if M has rank $t+1$, then the dimension of the moduli space of $M^{\vee}$ polarized K3 surfaces is $19-(18-t)$ which agrees with the rank of M. Returning to the question of Kähler deformations and complex deformations, we see that this definition of mirror symmetry matches the idea behind rotation of the Hodge diamond, as mentioned earlier.

2.2. Quasihomogeneous polynomials and diagonal symmetries.

We recall a few facts and definitions (cf. [**14**] for details). A *quasihomogeneous* map of degree d with integer weights $w_1, w_2, \ldots, w_n$ is $W : \mathbb{C}^n \to \mathbb{C}$ such that for every $\lambda \in \mathbb{C}$,

$$W(\lambda^{w_1} x_1, \lambda^{w_2} x_2, \ldots, \lambda^{w_n} x_n) = \lambda^d W(x_1, x_2, \ldots, x_n).$$

One can assume $\gcd(w_1, w_2, \ldots, w_n) = 1$ and say W has the *weight system* $(w_1, w_2, \ldots, w_n; d)$. Given a quasihomogeneous polynomial $W : \mathbb{C}^n \to \mathbb{C}$ with a critical point at the origin, we say it is *non-degenerate* if the origin is the only critical point of W and the fractional weights $w_1/d, \ldots, w_n/d$ of W are uniquely determined by W.

A non-degenerate quasihomogeneous polynomial W (also called *potential* in the literature) is *invertible* if it has the same number of monomials as variables.

If W is invertible we can rescale variables so that $W = \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{a_{ij}}$. This polynomial can be represented by the square matrix $A_W = (a_{ij})$, which we will call the *exponent matrix* of the polynomial. Since W is invertible, the matrix A_W is an invertible matrix.

The *group G_W of diagonal symmetries* of an invertible polynomial W is

$$G_W = \{(c_1, c_2, \ldots, c_n) \in (\mathbb{C}^*)^n : W(c_1 x_1, c_2 x_2, \ldots, c_n x_n) = W(x_1, x_2, \ldots, x_n)\}.$$

Observe that, given $\gamma = (c_1, c_2, \ldots, c_n) \in G_W$, the c_i's are roots of unity. Thus, one can consider G_W as a subgroup of $(\mathbb{Q}/\mathbb{Z})^n$, using additive notation and identifying $(c_1, c_2, \ldots, c_n) = (e^{2\pi i g_1}, e^{2\pi i g_2}, \ldots, e^{2\pi i g_n})$ with $(g_1, g_2, \ldots, g_n) \in (\mathbb{Q}/\mathbb{Z})^n$. Observe that the order of G_W is $|G_W| = \det(A_W)$.

Since W is quasihomogeneous, the *exponential grading operator* $j_W = (w_1/d, w_2/d, \ldots, w_n/d)$ is contained in G_W. We denote by J_W the cyclic group of order d generated by j_W: $J_W = \langle j_W \rangle$. Moreover, each $\gamma = (g_1, \ldots, g_n)$ define a diagonal matrix and thus G_W is embedded in $GL_n(\mathbb{C})$. We define

$$SL_W := G_W \cap SL(n, \mathbb{C}),$$

i.e., $\gamma = (g_1, \ldots, g_n) \in SL_W$ if and only if $\sum_i g_i \in \mathbb{Z}$. The group SL_W is called the *symplectic group* since, by [**3**, Proposition 1], an automorphism $\sigma \in G_W$ is symplectic if and only if $\det \sigma = 1$, that is, if and only if $\sigma \in SL_W$.

2.3. K3 surfaces from (W, G).

Reid (in an unpublished work) and Yonemura [**32**] have independently compiled a list of the 95 normalized weight systems $(w_1, w_2, w_3, w_4; d)$ ("the 95 families") such that $\mathbb{P}(w_1, w_2, w_3, w_4)$ admits a quasismooth hypersurface of degree d whose minimal resolution is a K3 surface. We consider one of these weight systems $(w_1, w_2, w_3, w_4; d)$ and an invertible quasihomogeneous polynomial of the form

$$W = x_1^m + f(x_2, x_3, x_4). \tag{1}$$

Moreover, let G be a group of symmetries such that $J_W \subseteq G \subseteq SL_W$ and let $\widetilde{G} = G/J_W$. The polynomial W defines a hypersurface $Y_{W,G} \subset \mathbb{P}(w_1, w_2, w_3, w_4)/\widetilde{G}$ and one shows that the minimal resolution $X_{W,G}$ of $Y_{W,G}$ is a K3 surface (see [**3**], [**14**]).

The group G_W acts on $Y_{W,G}$ via automorphisms, which extend to automorphisms on the K3 surface $X_{W,G}$. The given form of W ensures that the K3 surface $X_{W,G}$ admits a purely non-symplectic automorphism of order m:

$$\sigma_m : [x_1 : x_2 : x_3 : x_4] \mapsto [\zeta_m x_1 : x_2 : x_3 : x_4]$$

where ζ_m is a primitive m-th root of unity. With additive notation, it is $\sigma_m = (1/m, 0, 0, 0)$.

2.4. BHK mirror symmetry.

Now we can describe the second relevant formulation of mirror symmetry coming from mirror symmetry for Landau–Ginzburg models and which we call BHK (from Berglund–Hübsch–Krawitz) mirror symmetry. This particular formulation of mirror symmetry was developed initially by Berglund–Hübsch in [**10**], and later refined by Berglund–Henningson in [**9**] and Krawitz in [**23**]. Because of the LG/CY correspondence and a theorem from Chiodo–Ruan [**12**], this mirror symmetry of LG models can be translated into mirror symmetry for Calabi–Yau varieties (or orbifolds).

We consider (W, G) with W invertible and $W = \sum_{i=1}^n \prod_{j=1}^n x_j^{a_{ij}}$ and define another pair (W^T, G^T), called the BHK mirror. We first define the polynomial W^T as

$$W^T = \sum_{i=1}^n \prod_{j=1}^n x_j^{a_{ji}},$$

i.e., the matrix of exponents of W^T is A_W^T. By the classification of invertible polynomials (cf. [**24**, Theorem 1]), W^T is invertible.

Next, using additive notation, one defines the dual group G^T of G as

$$G^T = \left\{ g \in G_{W^T} \mid g A_W h^T \in \mathbb{Z} \text{ for all } h \in G \right\}. \tag{2}$$

The following useful properties of the dual group can be found in [**3**, Proposition 3]:

PROPOSITION 2.2 (cf. [**3**, Proposition 3]). *Given G and G^T as before, one has*:

1. $(G^T)^T = G$.
2. *If $G_1 \subset G_2$, then $G_2^T \subset G_1^T$ and $G_2/G_1 \cong G_1^T/G_2^T$.*
3. $(G_W)^T = \{0\}$, $(\{0\})^T = G_{W^T}$.
4. $(J_W)^T = SL_{W^T}$. *In particular, if $J_W \subset G$, then $G^T \subset SL_{W^T}$.*

Given the pair (W, G) with W invertible with respect to one of the 95 weight systems, we associated to it the K3 surface $X_{W,G}$. One can check that in this case the weight system of W^T also belongs to the 95. By the previous result, $J_{W^T} \subseteq G^T \subseteq SL_{W^T}$, so that X_{W^T,G^T} is again a K3 surface. We call X_{W^T,G^T} the *BHK mirror of* $X_{W,G}$.

2.5. Main theorem.

We have described two kinds of mirror symmetry for K3 surfaces: LPK3 mirror symmetry and BHK one. Since mirror symmetry describes a single physical phenomenon, we expect the two constructions to be compatible in situations where both apply. We will now state our main theorem, which shows that BHK and LPK3 mirror symmetry agree for the K3 surfaces $X_{W,G}$, when W is of the form (1). When no confusion arises, we will denote the mirror K3 surfaces $X_{W,G}$ and X_{W^T,G^T} simply by X and X^T.

Consider the data (W, G, σ_m), where

- W is an invertible polynomial of the form (1) whose weight system belong to the 95 families of Reid and Yonemura,
- $\sigma_m = (1/m, 0, 0, 0)$ is the non-symplectic automorphism of order m,
- G is a group of diagonal symmetries of W such that $J_W \subseteq G \subseteq SL_W$.

By Section 1.3, the invariant lattice $S_X(\sigma_m)$ is a primitive sublattice of S_X and $(X_{W,G}, S_X(\sigma_m))$ is a $S_X(\sigma_m)$-polarizable K3 surface. Let r be the rank of $S_X(\sigma_m)$. The BHK mirror is given by (W^T, G^T, σ_m^T), where σ_m^T is the non-symplectic automorphism of order m on X_{W^T,G^T}. Notice that σ_m and σ_m^T have the same form, namely $(1/m, 0, 0, 0)$, but they act on different surfaces.

THEOREM 2.3. *Suppose $m \neq 4, 8, 12$. If W is a polynomial of the form* (1)*, quasi-homogeneous with respect to one of the* 95 *weight systems for K*3 *surfaces as in Section* 2.3 *and G is a group of diagonal symmetries satisfying $J_W \subseteq G \subset SL_W$, then $\big(X_{W^T,G^T}, S_{X^T}(\sigma_m^T)\big)$ is an LPK*3 *mirror of $(X_{W,G}, S_X(\sigma_m))$.*

The theorem is proved by showing that

$$S_X(\sigma_m)^\vee \cong S_{X^T}(\sigma_m^T).$$

As we have seen in Section 1.2, this amounts to checking that the invariants $(r, q_{S_X(\sigma_m)})$ for $X_{W,G}$ and $(r^T, q_{S_{X^T}(\sigma_m^T)})$ for X_{W^T,G^T} satisfy $r = 20 - r^T$ and $q_{S_X(\sigma_m)} \cong -q_{S_{X^T}(\sigma_m^T)}$. Thus, the heart of the proof is determining $q_{S_X(\sigma_m)}$ (or equivalently in our case $S_X(\sigma_m)$). In the following section, we will describe how this is done. It involves computing the

invariant lattice and its overlattices. We list the results in tables in Section 5. Unfortunately, our method does not work for $m = 4, 8, 12$ due to the presence of many overlattices, so that we cannot exactly pinpoint the invariant lattice.

3. Methods.

In the setting of Theorem 2.3, one has to show that $S_X(\sigma_m)^\vee \cong S_{X^T}(\sigma_m^T)$. Whenever $m = p$ a prime number, Theorem 2.3 was proved using a similar method in [**3**] for $m = 2$ and [**14**] for other primes. There is not a general method of proof in either article; instead the theorem is checked in every case.

In [**3**] and [**14**] there are several tools introduced in order to facilitate computation of the invariant lattice. The proof we give here follows roughly the same idea, however the methods used in the previous articles for computing $S_X(\sigma_m)$ are no longer valid, when m is not prime. In order to illustrate the differences, we highlight briefly the method used in case of p prime. Then we will describe the proof of the theorem, in case m is not prime.

3.1. Method for $m = p$ prime.

As mentioned, the argument given in [**3**] and [**14**] essentially boils down to determining the invariant lattice $S_X(\sigma_p)$ for $X = X_{W,G}$. The method for determining this lattice relies on the following powerful theorems.

THEOREM 3.1 ([**6**]). *Given a K3 surface with a non-symplectic automorphism σ of order p, a prime, the invariant lattice $S_X(\sigma)$ is p-elementary, i.e., $A_{S_X(\sigma)} \cong (\mathbb{Z}/p\mathbb{Z})^a$.*

THEOREM 3.2 ([**27**], [**30**]). *For a prime $p \neq 2$, a hyperbolic, p-elementary lattice L with rank $r \geq 2$ is completely determined by the invariants (r, a), where a is the length of L. An indefinite 2-elementary lattice is determined by the invariants (r, a, δ), where $\delta \in 0, 1$ and $\delta = 0$ if the discriminant quadratic form takes values 0 or 1 only and $\delta = 1$ otherwise.*

By Proposition 1.1, the orthogonal complement in L_{K3} of a p-elementary lattice with invariants (r, a) is a p-elementary lattice with invariants $(22 - r, a)$. For both a given 2-elementary lattice and its orthogonal complement, the third invariant δ agrees. In the setting of Theorem 2.3, Corollary 1.5 shows that we also have $S_X(\sigma_p)^\perp \cong U \oplus M$ where M is a hyperbolic p-elementary lattice with invariants $(20 - r, a)$. Thus, for $p \neq 2$ it is enough to verify that (r, a) for $X_{W,G}$ and (r^T, a_T) for X_{W^T,G^T} satisfy $r = 20 - r^T$ and $a = a_T$. For $p = 2$ the third invariant δ must also be compared, which the authors checked in [**3**].

In order to compare (r, a), we first look at the topology of the fixed point locus.

THEOREM 3.3 (cf. [**6**], [**28**]). *Let X be a K3 surface with a non-symplectic automorphism σ of prime order $p \neq 2$. Then the fixed locus X^σ is nonempty, and consists of either isolated points or a disjoint union of smooth curves and isolated points of the following form*:

$$X^\sigma = C \cup R_1 \cup \cdots \cup R_k \cup \{p_1, \ldots, p_n\}. \tag{3}$$

Here C is a curve of genus $g \geq 0$, R_i are rational curves and p_i are isolated points.

If $p = 2$, then the fixed locus is either empty, the disjoint union of two elliptic curves, or is of the form (3) *with $n = 0$.*

In [**3**] and [**14**], σ_p always fixes a curve. Furthermore, the case of two elliptic curves does not appear in this setting described there. Therefore, the fixed locus is determined by the invariants (g, k, n). In [**6**], the authors give formulas to calculate (r, a) given (g, k, n) for each prime p (see [**6**, Theorem 0.1]). Thus, in order to prove Theorem 2.3 for p prime, one first computes the invariants (g, k, n), and from them computes the invariants (r, a) (if $p = 2$, additional computation are required to obtain δ). Then one compares the invariants for BHK mirrors as described above.

3.2. Method for m not prime.

If m is not prime, $S_X(\sigma_m)$ is no longer p-elementary for any prime p. The best we can say about $A_{S_X(\sigma)}$ is that it is a subgroup of $(\mathbb{Z}_m)^a$, as in the following lemma.

LEMMA 3.4. *Let σ be a purely non-symplectic automorphism of order m. The discriminant group $A_{S_X(\sigma)}$ is a subgroup of $(\mathbb{Z}_m)^a$ for some $a \geq 0$.*

PROOF. We know $A_{S_X(\sigma)}$ is a finite abelian group. Thus, it suffices to show that for each $x \in A_{S_X(\sigma)}$, we have $mx = 0$. By Maschke's theorem, the vector space $H^2(X, \mathbb{Z}) \otimes \mathbb{Q}$ is completely reducible. Therefore, it decomposes as a sum of irreducible $\mathbb{Q}[\mathbb{Z}/m\mathbb{Z}]$-modules. Each direct summand is of the form $\mathbb{Q}(\xi_d)$ where $d|m$. So we have $\sum_{i=0}^{m-1}(\sigma^*)^i = 0$ on $T_X(\sigma)$. Now we have $A_{S_X(\sigma)} \cong H^2(X, \mathbb{Z})/(S_X(\sigma) \oplus T_X(\sigma)) \cong A_{T_X(\sigma)}$, and these automorphisms commute with the action of σ^*. Thus, σ^* acts by identity on $A_{T_X(\sigma)}$, but since $\sum_{i=0}^{m-1}(\sigma^*)^i = 0$ on $T_X(\sigma)$, $mx = 0 \pmod{T_X(\sigma)}$ for $x \in T_X(\sigma)^*$. □

We see that $A_{S_X(\sigma_m)}$ can have any possible order dividing a power of m. Even in the best of situations (e.g., $A_{S_X(\sigma_m)} \cong \mathbb{Z}_m^a$) the lattice is no longer determined by (r, a). Consider, for example, the two lattices $L_1 = U \oplus D_5^2$ and $L_2 = U \oplus A_3 \oplus D_7$: they both have discriminant group $\mathbb{Z}_4^2$ and $(r, a) = (12, 2)$, but $q_{L_1} = (w_{2,2}^3)^2$ whereas $q_{L_2} = w_{2,2}^1 \oplus w_{2,2}^5$. In other words, we have two *different* lattices with the same rank, and isomorphic discriminant groups (though different discriminant quadratic *forms*).

It is true, however, that the lattice is determined in all cases considered here by the rank and the discriminant quadratic form (cf. Corollary 1.4). Thus, rather than computing (r, a) it suffices to compute $(r, q_{S_X(\sigma_m)})$. We will be able to compute these invariants for all K3 surfaces $X_{W,G}$ with W as in (1) and $G \subset SL_W$ except in case $m = 4, 8, 12$.

The first step to compute $S_X(\sigma_m)$ is to compute its rank r. In order to do so, we describe an important set of curves on $X_{W,G}$, namely the orbits of the exceptional curves. Consider the hypersurface $Y_{W,G}$ as in Section 2.3, and the resolution of singularities $X_{W,G} \to Y_{W,G}$. Because W is nondegenerate, all of the singularities of $Y_{W,G}$ lie on the coordinate curves.

Let $\mathcal{E}$ denote the set of the exceptional curves on $X_{W,G}$. Because σ_m leaves the coordinate curves invariant, the action of σ_m on $X_{W,G}$ induces an action of σ_m on $\mathcal{E}$. Given an exceptional curve E, let G_E denote the isotropy group for E and let $\mathcal{E}/\sigma_m$

denote the set of orbits of this action. Now we set

$$b_E = \frac{1}{|G_E|} \sum_{i=0}^{m-1} \sigma_m^i E.$$

This is simply the sum of all curves in the orbit of E. In particular, if E is fixed by σ_m, then $b_E = E$. We can use this topology to determine r:

LEMMA 3.5. *The rank r of $S_X(\sigma)$ is equal to 1 plus the number of orbits of exceptional curves in the blow-up $X_{W,G} \to Y_{W,G}$, i.e.,*

$$\operatorname{rank} S_X(\sigma_m) = 1 + |\mathcal{E}/\sigma_m|.$$

PROOF. Consider the quotient $X_{W,G}/\sigma_m$. We have the following diagram

$$\begin{array}{ccc} X_{W,G} & \longrightarrow & Y_{W,G} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ X_{W,G}/\sigma_m & \longrightarrow & \mathbb{P}(w_2, w_3, w_4) \end{array}$$

where the bottom horizontal arrow is obtained by blowing up isolated singular points of $\mathbb{P}(w_2, w_3, w_4)$ and singular points on the branch locus $\{f(x_2, x_3, x_4) = 0\} \subset \mathbb{P}(w_2, w_3, w_4)$. Notice in the quotient $\mathbb{P}(w_2, w_3, w_4) \cong Y_{W,G}/\sigma_m$, the branch locus corresponds to the curve $x_1 = 0$ in $Y_{W,G}$. Furthermore, each orbit of exceptional curves will correspond to exactly one exceptional curve of the bottom horizontal map.

Since $h^2(\mathbb{P}(w_2, w_3, w_4)) = 1$ (see e.g., [**21**] or [**18**]), and $H^2(X_{W,G}/\sigma_m, \mathbb{Q})$ is generated by the exceptional curves and the pullback from $\mathbb{P}(w_2, w_3, w_4)$, we have

$$\dim H^2(X_{W,G}/\sigma_m, \mathbb{Q}) = 1 + |\mathcal{E}/\sigma_m|.$$

Finally, we know by [**26**] that $H^2(X_{W,G}/\sigma_m, \mathbb{Q}) = H^2(X_{W,G}, \mathbb{Q})^{\sigma_m}$. This implies the result. □

REMARK. Although $H^2(X_{W,G}, \mathbb{Q})^{\sigma_m}$ is generated by the orbits of the exceptional curves and one coordinate curve, this is not always the case for the lattice $S_X(\sigma_m)$.

The second step in determining (r, q) requires another set of curves. Let $\mathcal{K}$ be the set of smooth irreducible components of (the strict transforms of) the coordinate curves $x_i = 0$ (we omit curves that have worse singularities than separating nodes between two or more smooth components). As before σ_m acts on this set. For $C \in \mathcal{K}$, we define G_C as the isotropy group, $\mathcal{K}/\sigma_m$ as the set of orbits, and

$$b_C = \frac{1}{|G_C|} \sum_{i=0}^{m-1} \sigma_m^i C.$$

Notice that σ_m leaves the coordinate curves invariant, so b_C is the sum of some of the smooth irreducible components of one of the coordinate curves. In cases where the

coordinate curve is smooth, b_C is the coordinate curve. There are cases, however, where the coordinate curve is not irreducible.

Finally we define the set of orbit curves $\mathcal{B} = \{b_E \mid E \in \mathcal{E}/\sigma_m\} \cup \{b_C \mid C \in \mathcal{K}/\sigma_m\}$. Let $L_{\mathcal{B}}$ denote the sublattice of S_X generated by $\mathcal{B}$. This lattice is clearly a sublattice of $S(\sigma_m)$ since σ_m fixes each of the generators. Therefore, $\operatorname{rank} L_{\mathcal{B}} \leq r$. By Lemma 3.5, we see that $\operatorname{rank} L_{\mathcal{B}} = r$. Hence $S(\sigma_m)$ is an overlattice of $L_{\mathcal{B}}$. Because of this explicit description, we can actually determine the lattice $L_{\mathcal{B}}$ in each case. (See the appendix for Magma code, which allows us to compute $L_{\mathcal{B}}$.)

Determining (r, q) now breaks into four different cases, each of which requires a different method. We now describe those four methods. In most cases (except for Method II), we show $L_{\mathcal{B}} = S_X(\sigma_m)$.

3.2.1. Method I.

The first method is the most common. Using Proposition 1.6, we can determine all of the overlattices of $L_{\mathcal{B}}$. It often happens that the discriminant group has no isotropic subgroups, and so there are no nontrivial overlattices. Since $S_X(\sigma_m)$ is an overlattice of $L_{\mathcal{B}}$, we can conclude $S_X(\sigma_m) = L_{\mathcal{B}}$. An example of this situation is shown in Example 4.1.

This covers all possibilities when $m = 18, 20, 24, 30, 42$, and parts of the other orders, namely whenever m and r are one of the following possibilities: $m = 10$, $r = 3, 8, 12, 17$; $m = 9$, $r = 4, 16$; and $m = 6$, $r = 1, 4, 6, 9, 11, 14, 16, 19$.

3.2.2. Method II.

The second method does not use $L_{\mathcal{B}}$. Instead, we notice that some power σ_m^k has order p for some prime p and $S_X(\sigma_m) \subset S_X(\sigma_m^k)$ is a primitive embedding of lattices. If it happens that $\operatorname{rank} S_X(\sigma_m) = \operatorname{rank} S_X(\sigma_m^k)$, then indeed $S_X(\sigma_m) = S_X(\sigma_m^k)$. In this case $S_X(\sigma_m)$ is p-elementary and we can use the methods in Section 3.1 to compute it. See Example 4.2 for an example of this situation.

This covers the remaining cases with $m = 6, 10, 14, 15, 22$ and the case with $m = 9$, $r = 16$. That leaves exactly four cases, namely $m = 9, r = 8, 12$ and both cases with $m = 16$.

3.2.3. Method III.

The third method is for $m = 9$, $r = 8$ and $m = 16$, $r = 9$. For this method, we consider the family of K3 surfaces of Belcastro in [**8**]. Belcastro has computed the Picard lattice of the general member of a family of K3 surfaces. Each invertible polynomial W that we consider in this article yields a special member of such a family, but often with bigger Picard lattice. We will use the Picard lattice of the general member of the family to show that $L_{\mathcal{B}}$ is primitively embedded into $S_X(\sigma_m)$.

To do this, we consider the family of K3 surfaces defined by polynomials quasihomogeneous with respect to each of the weight systems $(3, 3, 2, 1; 9)$ and $(8, 4, 3, 1; 16)$, respectively (these weight systems are numbered 18 and 37, resp.). Let X_η denote the general member of this family and let $X = X_{W,G}$. In the construction of each family in question, one must resolve the singularities of a family of hypersurfaces in $\mathbb{P}(w_1, \ldots, w_4)$. In doing so, we see that the same generators of $L_{\mathcal{B}}$ also generate S_{X_η}, giving us a canonical isomorphism

$$L_{\mathcal{B}} \xrightarrow{\cong} S_X \ .$$

In [**19**, Remark 2.13], Huybrechts showed, that since X is a special member of this family, we get an primitive embedding

$$\mathrm{sp} : S_X \ \hookrightarrow S_X$$

making the following diagram commute

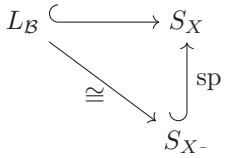

Figure 1.

Therefore, $S_X/L_{\mathcal{B}} \cong S_X/S_X$, and since the latter is a free abelian group, the lattice $L_{\mathcal{B}}$ is a primitive sublattice of S_X. Since $S_X(\sigma_m)/L_{\mathcal{B}} \subset S_X/L_{\mathcal{B}}$ and the latter is a free abelian group, we see that $L_{\mathcal{B}}$ is a primitive sublattice of $S_X(\sigma_m)$. Since they have the same rank, we have $L_{\mathcal{B}} = S_X(\sigma_m)$. We illustrate this method in Example 4.3.

3.2.4. Method IV.

This leaves two cases: $m = 16$, $r = 11$ and $m = 9$, $r = 12$. As each of these cases is different we will work out each case in Examples 4.5 and 4.6. This method involves finding $S_{X_{W,G}}$ and showing explicitly that $L_{\mathcal{B}}$ embeds primitively into $S_{X_{W,G}}$. Then, as in Method III, we can conclude that $L_{\mathcal{B}} = S_X(\sigma_m)$.

Remark. What we see when employing these four methods is that the invariant lattice seems to be the lattice $L_{\mathcal{B}}$. This is certainly true for every invariant lattice computed by Methods I, III and IV, though it also seems to be true for each of the others (including the invariant lattice for the automorphisms of prime order and of order 4, 8, 12), though we have not checked every case explicitly.

In what follows, we will make use of the following lemma.

Lemma 3.6 (cf. [**5**], [**15**]). *Let $R_1, R_2, \ldots, R_s$ be a tree of smooth rational curves on a K3 surface X and σ a non-symplectic automorphism of finite order on X leaving each of the R_i invariant. Then the intersection points of the R_i's are fixed by the automorphism and it suffices to know the action at one intersection point to know the action on the entire tree.*

We will also make use of the following formula, giving the genus of a curve of degree d in weighted projective space $\mathbb{P}(w_1, w_2, w_3)$ in order to compute the genus of the coordinate curves. This formula can be found, e.g., in [**20**, Theorem 12.2].

$$g(C) = \frac{1}{2}\left(\frac{d^2}{w_1w_2w_3} - d\sum_{1\le j<i}\frac{\gcd(w_i,w_j)}{w_iw_j} + \sum_{i=1}^{3}\frac{\gcd(w_i,d)}{w_i} - 1\right).$$

With the genus, one can also compute self-intersection numbers for curves on a K3 surface via Riemann–Roch:

$$2g(C) - 2 = C \cdot C.$$

3.3. Proof of main theorem.

We now provide the details of the proof of Theorem 2.3. For each K3 surface $X_{W,G}$ we have the invariants $(r, q_{S_X(\sigma_m)})$ for the invariant lattice $S_X(\sigma_m)$, as discussed in the previous section. We know the invariant lattice has signature $(1, r-1)$, and so the orthogonal complement has signature $(2, 20-r)$. We check that the conditions of Corollary 1.5 are fulfilled so that $S_X(\sigma_m)^{\perp}_{L_{\mathrm{K3}}} \cong U \oplus M$ for some lattice M.[2] This lattice M is hyperbolic with signature $(1, 19-r)$ and has discriminant quadratic form $-q_{S_X(\sigma_m)}$. One can see from the tables that the conditions of Theorem 1.3 are satisfied. Hence, there is exactly one lattice with these invariants. To prove the theorem, therefore, we need simply to check that $(20-r, -q_{S_X(\sigma_m)})$ are the invariants for the invariant lattice $S_{X^T}(\sigma_m^T)$ for X_{W^T,G^T}. This can be checked by consulting the tables in Section 5. This concludes the proof.

Tables 2–13 contain all possible invertible polynomial of the form (1) with non-symplectic automorphism of order m, and for each polynomial, we list the orders of the possible groups G/J_W satisfying $J_W \subset G \subset SL_W$. In most cases SL_W/J_W is cyclic, so the properties of G^T make it clear what the dual group is. However, for two examples, one can see by the multiplicity of subgroups with the same order that the group SL_W/J_W is not cyclic. These two examples are $x^3+y^3+z^6+w^6$ (number 3d) and $x^2+y^4+z^6+w^{12}$ (number 8d) in Table 13. In these cases, we will clear up any ambiguities in the following sections.

From now on we will make a change of notation from (x_1, x_2, x_3, x_4) to (x, y, z, w) for the variables of W, so that the variables are arranged with the weights in nonincreasing order. In other words, it is possible that x_1 corresponds to any of x, y, z, or w. This convention is also used in tables of Section 5.

REMARK. It is possible that a given K3 surface admits a purely non-symplectic automorphism of different orders. It turns out that it does not matter which automorphism one uses to exhibit LPK3 mirror symmetry, the notion still agrees with BHK mirror symmetry, in the sense of Theorem 2.3, as long as the defining polynomial is of the proper form (1). We expect that the theorem still holds for K3 surfaces that do not take the form of (1), but that is a topic for further investigation.

[2]In one case, the conditions are not fulfilled, namely $m = 6$, $r = 1$. However, in this case, we know the lattice $U \oplus \langle 4\rangle$ has the given invariants, and by Corollary 1.4 this is the only such lattice.

4. Examples.

In this section we will first give examples to illustrate each of the methods that was used to determine $S_X(\sigma_m)$. Then we will describe the subgroups in the two cases where SL_W/J_W is not cyclic.

Example 4.1. Method I:

This first example will illustrate Method I for determining $S_X(\sigma_m)$. Let us consider the K3 surface with equation

$$W = x^2 + y^3 + z^9 + yw^{12} = 0$$

in the weighted projective space $\mathbb{P}(9, 6, 2, 1)$ with degree 18. This is number 12b in Table 12. There are two non-symplectic automorphisms of interest $\sigma_2 : (x, y, z, w) \mapsto (-x, y, z, w)$ and $\sigma_9 : (x, y, z, w) \mapsto (x, y, \mu_9 z, w)$. The invariant lattice $S_X(\sigma_2)$ was dealt with in [**3**], so we focus on σ_9.

Here $|G_W| = 36 \cdot 18$, $|J_W| = 18$ and the weight system for the BHK mirror is $(18, 11, 4, 3; 36)$ so that $[G_W : SL_W] = 36$. Thus, $|SL_W/J_W| = 1$.

Looking at the action of $\mathbb{C}^*$ on the weighted projective space $\mathbb{P}(9, 6, 2, 1)$, we find the following isotropy:

$$\mu_3 : \text{fixes } z = w = 0,\ x^2 + y^3 = 0,$$
$$\mu_2 : \text{fixes } x = w = 0,\ y^3 + z^9 = 0.$$

The first row provides a single point with $\mathbb{Z}/3\mathbb{Z}$ isotropy (A_2 singularity), and the second provides three points each with $\mathbb{Z}/2\mathbb{Z}$ isotropy (3 A_1's). Their resolution gives the configuration of curves on $X_{W,G}$ depicted in Figure 2. In this depiction, we have not indicated the three intersection points between C_x and C_z.

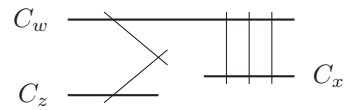

Figure 2. Resolution of singularities on X_W.

The set $\mathcal{E}$ consists of these five exceptional curves. Denote by E_1, E_2 and E_3 the three A_1 fibers, and E_4 and E_5 the two curves in the A_2 fiber. Looking at the form of W, we see that the curves $C_x = \{x = 0\}$, $C_z = \{z = 0\}$ and $C_w = \{w = 0\}$ are smooth. The curve $C_y = \{y = 0\}$ is not smooth. Thus, the set $\mathcal{K}$ consists of these three smooth curves. The curve C_x has genus 7, C_z has genus 1, and C_w has genus 0.

There are two important representatives of the coset $\sigma_9 J_W$ in G_W which will help us compute the fixed locus for σ_9, namely $(0, 0, 1/9, 0)$ and $(0, 2/3, 0, 4/9)$. These representatives show us that the curve $C_z = \{z = 0\}$ is fixed and the point defined by $\{w = y = 0,\ x^2 + z^9 = 0\}$ is also fixed. Because C_z is fixed, E_4 and E_5 are invariant (though not fixed pointwisely). The point of intersection of C_w and the A_2 exceptional

fiber and this other point are the only fixed points on C_w. Thus, the three A_1 singularities are permuted by the action. By Lemma 3.5, since there are three orbits, $S_X(\sigma_9)$ has rank 4.

We now compute the lattice $L_{\mathcal{B}}$, generated by $\mathcal{B} = \{E_1+E_2+E_3, E_4, E_5, C_x, C_z, C_w\}$. Since there are six generators, two of them are redundant, for example C_x and C_z. Consider the lattice L generated by $E_1 + E_2 + E_3$, E_4, E_5, and C_w. This lattice has bilinear form

$$\begin{pmatrix} -6 & 0 & 0 & 3 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 3 & 0 & 1 & -2 \end{pmatrix}$$

which has discriminant form $\omega_{3,1}^1$. By Proposition 1.6, there are no non-trivial even overlattices of this lattice, hence $L = L_{\mathcal{B}} = S_X(\sigma_9)$. Thus, we have the invariants $(r, q_{S_X(\sigma_9)}) = (4, w_{3,1}^1)$. In fact, $S_X(\sigma_9) \cong U \oplus A_2$.

REMARK. This method also yields some other interesting facts regarding the Picard lattice of these surfaces. In [**8**], Belcastro computes the Picard lattice for a generic hypersurface with these weights and degree as $U \oplus D_4$. However, if we look at the non-symplectic automorphism σ_9^3, we can compute the invariants $g = 1, n = 4, k = 1$, and therefore $r = 10, a = 4$ for the invariant lattice, giving us the invariant lattice $S_X(\sigma_9^3) = U \oplus A_2 \oplus E_6$. This shows us in particular, that the Picard lattice of this surface is bigger than the Picard lattice for a generic quasihomogeneous polynomial with these weights.

EXAMPLE 4.2. Method II:

In order to illustrate Method II, we repeat the computations for the BHK mirror of the previous example:

$$W^T = x^2 + y^3w + z^9 + w^{12}$$

with weight system $(18, 11, 4, 3; 36)$. This is number 43a in Table 12. Here again $|SL_W/J_W| = 1$. As before, we also have an involution, but we consider only σ_9^T.

Looking at the action of $\mathbb{C}^*$ on $\mathbb{C}^4$ and resolving the singularities we have an A_{10} given by resolving the point $(0, 1, 0, 0)$, $2A_2$ coming from the two points with $y = z = 0$ fixed by μ_3 and an A_1 coming from the point with $y = w = 0$ fixed by μ_2. This time $\mathcal{E}$ has 15 curves and $\mathcal{K} = \{C_x, C_y, C_z\}$ as in Figure 3. Again we do not depict the three points of intersection between C_x and C_y.

Two relevant representatives of σ_9^T in G_{W^T}/J_{W^T} are $(0, 0, 1/9, 0)$ and $(0, 4/9, 0, 2/3)$. From these we see that the curve C_z is fixed. It has genus 0. Furthermore, the exceptional curve from the A_1 singularity at $y = w = 0$ is fixed pointwisely, as well as one of the curves in the exceptional A_{10}. Since C_z is fixed, each of the exceptional curves is left invariant under σ_9^T. From Lemma 3.5, the rank of the invariant lattice $S_{X^T}(\sigma_9^T)$ is $r = 16$.

In this case, we compute the invariant lattice for $(\sigma_9^T)^3$, which is a non-symplectic automorphism of order 3. The curves C_z and C_y are fixed and have genus 0 and 1,

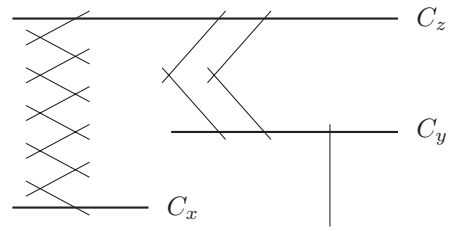

Figure 3. Resolution of singularities on X_{W^T}.

respectively. Three of the curves on the A_{10} chain are also fixed, as in Lemma 3.6. Furthermore, the remaining intersection points of the chains of exceptional curves are fixed, and an additional point on the A_1. Thus, the invariants are $(g, k, n) = (1, 4, 7)$. Using the results cited in Section 3.1, the invariants for the 3-elementary lattice $(\sigma_9^T)^3$ are $(16, 1)$. Since $S_{X^T}(\sigma_9^T)$ is a primitive sublattice of this 3-elementary lattice, and both have the same rank, they are equal. Therefore, we have invariants $(r, q) = (16, w_{3,1}^{-1})$ and the lattice is $S_X(\sigma_9) = U \oplus E_6 \oplus E_8$.

Comparing the ranks, and noticing that $\omega_{3,1}^1 = -\omega_{3,1}^{-1}$, we see the BHK mirror matches the LPK3 mirror symmetry.

Example 4.3. Method III:

Let $W := x^2 + y^4 + yz^4 + w^{16}$ with $m = 16$ in weight system $(8, 4, 3, 1; 16)$. This is number 37b in Table 8. The order of SL_W/J_W is 2. This appears to be the same K3 surface investigated in [**2**, Example 3.2]. Computing singularities we obtain an A_2 at the point $(0 : 0 : 1 : 0)$ and two A_3's at the two points with $z = w = 0$. Resolving these, we obtain the configuration of curves showed in Figure 4. The curves C_x and C_z intersect in four points, which are not depicted. The genus of the curve C_w is 0, the genus of C_z is 1, the genus of C_x is 6. However, C_y consists of two components, each a copy of $\mathbb{P}^1$.

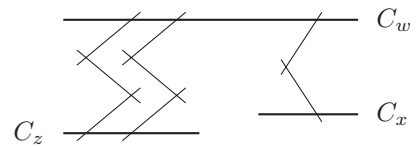

Figure 4. Resolution of singularities for X_W.

The automorphism $\sigma_{16} = (0, 0, 0, 1/16)$ fixes C_w, and therefore leaves all of the exceptional curves invariant. Thus, we have $|\mathcal{E}/\sigma_{16}| = 8$, and $r = 9$. Furthermore, $\mathcal{K}$ consists of the curves C_w, C_z and the two curves that make up C_y. Using an explicit form of the intersection matrix, one can check that the lattice $L_{\mathcal{B}}$ is actually generated by the exceptional curves and C_w. This is a lattice of type $T_{3,4,4}$. The discriminant group of $T_{3,4,4}$ is $\mathbb{Z}/8\mathbb{Z}$ and the corresponding form q is $w_{2,3}^5$. This form has one overlattice, and so we cannot use Method I. Method II will also not work here.

In [**8**], Belcastro computed the Picard lattice for a general member of the family of K3 surfaces with this weight system, using the same configuration of curves described above, giving us the canonical map $L_{\mathcal{B}} \cong T_{3,4,4}$ that fits into the diagram in Figure 1. Thus, $L_{\mathcal{B}}$ embeds primitively into $S_X(\sigma_m)$, and so they are equal, i.e., $S_X(\sigma_{16}) = L_{\mathcal{B}}$ with invariants $(r, q) = (9, w_{2,3}^5)$.

REMARK. There is another case with the same invariant lattice in the same weight system, namely number 37a. The reasoning is similar to what we have just outlined.

Finally, we will describe both of the cases requiring what we have called Method IV. These two cases are similar in that we use the Picard lattice to help determine $S_X(\sigma_m)$. We will need the following proposition.

PROPOSITION 4.4 ([**27**, Proposition 1.15.1]). *The primitive embeddings of a lattice L into an even lattice with invariants (m_+, m_-, q) are determined by the sets $(H_L, H_q, \gamma; K, \gamma_K)$, where $H_L \subset A_L$ and $H_q \subset A_q$ are subgroups, $\gamma : q_L|_{H_L} \to q|_{H_q}$ is an isomorphism of subgroups preserving the quadratic forms to these subgroups, K is an even lattice with invariants $(m_+ - t_+, m_- - t_-, -\delta)$, where $\delta \cong q_L \oplus (-q)|_{\Gamma_\gamma^\perp/\Gamma_\gamma}$, Γ_γ being the pushout of γ in $A_L \oplus A_q$, and, finally, $\gamma_K : q_K \to (-\delta)$ is an isomorphism of quadratic forms.*

From this proposition, we can determine all primitive embeddings of one even lattice into another. We will use this in the next example.

EXAMPLE 4.5. Method IV:

We now consider the BHK dual to the previous example. This is the first entry for 37b in Table 8. As mentioned in the introduction, this provides an example to the case in [**2**], where no example could be found. In this case, we have

$$W^T = W = x^2 + y^4 + yz^4 + w^{16},$$

and $G^T = SL_W$, and we know $|SL_W/J_W| = 2$. In fact the group is generated by $(1/2, 0, 1/2, 0)$, so we see that the points with $x = z = 0$ are fixed. Another representative in the same coset is $(1/2, 0, 0, 1/2)$. Thus, we see that the intersection points on the A_2 chain from the previous example are fixed, as well as other point with $x = w = 0$. The two A_3 chains are permuted by the action. Thus, on X_{W^T,G^T} we get the configuration of curves of Figure 5.

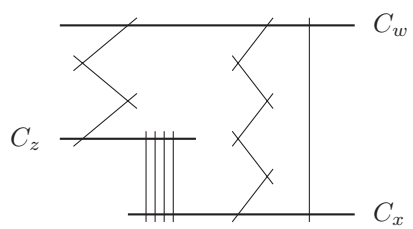

Figure 5. Resolution of singularities on X_{W^T,G^T}.

Using the Riemann–Hurwitz theorem, we can compute the genus of the coordinate curves. The curve C_x is covered by a curve of genus 6 with 6 fixed points, so it has genus 2. Similarly, we see that the genus of C_w is 0 and the genus of C_z is 0. The two components of the curve $\{y = 0\}$ from the previous example are permuted, to give us C_y of genus 0.

The non-symplectic automorphism $\sigma_{16} = (0, 0, 0, 1/16)$, fixes C_w, and therefore the chains of exceptional curves intersecting C_w are invariant. It is not difficult to see also that the four exceptional curves intersecting C_x and C_z are permuted. Thus, $r = 11$. One can check that C_z, C_y and C_x are superfluous, giving us a lattice $L_{\mathcal{B}}$ with discriminant form $w_{2,3}^{-5}$.

There is one isotropic subgroup H and hence one overlattice of $L_{\mathcal{B}}$. By Proposition 1.6 this overlattice has discriminant form $\omega_{2,1}^{-1}$. Since $S_X(\sigma_{16})$ is an overlattice of $L_{\mathcal{B}}$, the two possibilities for $S_X(\sigma_{16})$ are $U \oplus E_8 \oplus A_1$ or $T_{2,5,6}$. Using Proposition 4.4 we will show that $U \oplus E_8 \oplus A_1$ does not embed primitively into $S_{X_{W^T,G^T}}$, so that $S_X(\sigma_{16}) = T_{2,5,6}$.

In [**2**] Al Tabbaa–Sarti–Taki have computed the Picard lattice for K3 surfaces with non-symplectic automorphisms of order 16, and found that in our case, the Picard lattice is $U(2) \oplus D_4 \oplus E_8$. This lattice is 2-elementary with $u \oplus v$ as discriminant quadratic form. In particular, this quadratic form takes values 0 or 1 (i.e., $\delta = 0$).

On the other hand, $\omega_{2,1}^{-1}$ has value $3/2$ on the generator for $\mathbb{Z}/2\mathbb{Z}$. By Proposition 4.4, a primitive embedding of $U \oplus E_8 \oplus A_1$ into the Picard lattice $U(2) \oplus D_4 \oplus E_8$ must therefore correspond to the trivial subgroup. The existence of such a primitive embedding depends on the existence of an even lattice with invariants $(0, 3, u \oplus v \oplus \omega_{2,1}^1)$. The length of this discriminant quadratic form is 5, whereas the rank of the desired lattice is 3, and so no such lattice exists (see [**27**, Theorem 1.10.1]).

We conclude that the invariant lattice is $S_X(\sigma_{16}) \cong T_{2,5,6}$, which has invariants $(11, \omega_{2,3}^{-5})$.

Remark. The other case with $m = 16$, $r = 11$ is number 58 in Table 8. The method for computing the invariant lattice in this case is very similar to what we have just computed. Alternatively, that case can also be computed with Method III.

Example 4.6. Method IV:

The other case that requires Method IV is $m = 9$, $r = 12$. This occurs for two of the K3 surfaces, namely 18a and 18b, both instances using the group SL_W. Both of these cases are similar, so we describe only the first.

Using methods similar to those described in the previous examples, we get the configuration of curves depicted in Figure 6. For the discussion that follows, we denote by E_1 the exceptional curve in the A_5 chain, which intersects C_x.

The automorphism σ_9 permutes the three A_2 chains (yielding one orbit for each curve in the chain for a total of 2 orbits), but leaves the other nine exceptional curves, as well as the coordinate curves, invariant. This gives us $r = 12$, and we can compute the lattice $L_{\mathcal{B}} \cong M_9 \oplus A_2 \oplus E_8$ with discriminant quadratic form $w_{3,1}^1 \oplus w_{3,2}^1$. There is one isotropic subgroup of this lattice, corresponding to the overlattice $U \oplus A_2 \oplus E_8$. So we must find some way to show that $L_{\mathcal{B}}$ is primitively embedded in the Picard lattice, for then it is the invariant lattice.

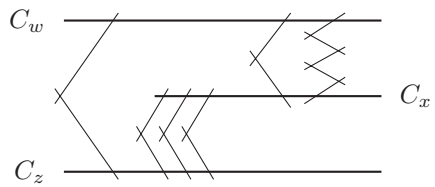

Figure 6. Resolution of curves on $X_{W,G}$ for example 18a.

We can determine the Picard lattice $S_{X_{W,G}}$. We first notice that σ_9^3 has order 3. Furthermore, its fixed locus has invariants $(g,k,n) = (0,3,7)$. Therefore its invariant lattice $S_X(\sigma_9^3)$ is a 3-elementary lattice with invariants $(16,3)$, i.e., it is the lattice $U \oplus E_8 \oplus 3A_2$, with discriminant quadratic form $3w_{3,1}^1$. Since the transcendental lattice $S_{X_{W,G}}^{\perp}$ has rank divisible by $\phi(9) = 6$, $S_X(\sigma_9^3)$ is the Picard lattice.

In fact, we will determine a basis for $S_{X_{W,G}}$. Consider the set $\mathcal{E}_1$ consisting of all of the exceptional curves (15 of them), and $\mathcal{K}$ consisting of all irreducible components of (the strict transforms of) the coordinate curves. Let $\mathcal{B}_1 = \mathcal{E}_1 \cup \mathcal{K}$. This set generates $S_{X_{W,G}}$. One can check by direct computation ([**11**]) that C_x and E_1 are redundant.

Now we consider the set $\mathcal{B}$, generating the lattice $L_{\mathcal{B}}$. Again, we compute that C_x and E_1 are redundant, so we get $L_{\mathcal{B}}$ generated by the two orbits from the A_2 chains, the remaining exceptional curves, and C_w and C_z. Two of the generators for $L_{\mathcal{B}}$ are just sums of generators of $S_{X_{W,G}}$. Thus, a change of basis shows that $S_X/L_{\mathcal{B}}$ is a free group of rank 4, and so $L_{\mathcal{B}}$ is primitively embedded.

The other example is similar. Instead of three A_2 chains, there are three A_1's. One can check that the set $\{y = 0\}$ is composed of three curves, each of genus zero. Each of these curves intersects one of the A_1 curves. These are permuted by the action of σ_9. Up to a relabelling, we obtain the same configuration of curves, and the same Picard lattice.

The only cases where SL_W/J_W is not cyclic are $x^3+y^3+z^6+w^6$ (number 3d) and $x^2+y^4+z^6+w^{12}$ (number 8d) in Table 13. We analyze them separately.

EXAMPLE 4.7. The first polynomial we consider is $W = x^3+y^3+z^6+w^6$ in $\mathbb{P}(2,2,1,1)$. This is number 3d in Table 13.

In this case the order of SL_W/J_W is 9 and it results that $SL_W/J_W = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ since there are no elements of order 9. The group J_W is generated by $j_W = (1/2,1/3,1/6,1/6)$ and two generators for SL_W/J_W are $g_1 = (1/3,2/3,0,0)$ and $g_2 = (1/3,1/3,1/3,0)$. Also we name $g_3 = (1/3,0,2/3,0)$ and $g_4 = (0,1/3,0,2/3)$. There are four subgroups of SL_W/J_W of order 3, namely $G_i = \langle g_i, j_W\rangle$, $i = 1,2,3,4$. We can also observe that $G_1^T = G_2$, $G_3^T = G_3$ and $G_4^T = G_4$.

Now we consider the non-symplectic automorphism $\sigma_6 = (0,0,0,1/6)$. One may notice, there is another automorphism of order 6, namely $(0,0,1/6,0)$, but due to symmetry (i.e., exchanging z and w), we must only consider one of them.

Using the same methods as before for each group, we can compute the invariant lattice for the corresponding K3 surface. In each case, the lattice $L_{\mathcal{B}}$ has no overlattices, so we can use Method I. When $G = G_3$, the invariant lattice has $(r,q) = (10, 4w_{3,1}^1)$, which is self-dual. The same is true for $G = G_4$.

When $G = G_1$ we get an invariant lattice with rank 16 and the discriminant form is $v \oplus w_{3,1}^1$. This is the dual of the invariant lattice we get with the choice $G = G_2$ and so it proves the theorem for this case.

EXAMPLE 4.8. Finally, we examine the polynomial $W = x^2 + y^4 + z^6 + w^{12}$ in $\mathbb{P}(6,3,2,1)$. This is number 8d in Table 13. There are non-symplectic automorphisms of order 2, 4 and 12, but we again focus on the non-symplectic automorphism of order 6: $\sigma_6 = (0,0,1/6,0)$.

The order of SL_W/J_W is 4 and since there are no elements of order 4, we conclude that $SL_W/J_W = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. The elements

$$g_1 = \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right),\ g_2 = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right),\ g_3 = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$$

each have order 2 and represent different cosets in SL_W/J_W; let $G_i := \langle g_i, j_W \rangle,\ i = 1,2,3$. Observe that $G_1^T = G_1$ while $G_2^T = G_3$.

When $G = G_1$, with Method I we compute the invariant lattice and obtain $(r,q) = (10, v \oplus 4w_{2,1}^{-1})$.

As for G_2, we use again Method I and obtain $(r,q) = (14, 2w_{2,1}^{-1} \oplus w_{3,1}^1)$, while for G_3 we get $(r,q) = (6, 2w_{2,1}^1 \oplus w_{3,1}^{-1})$. Observing they are mirror of each other, we can conclude that the theorem is proved in this case.

5. Tables.

In each table, we have arranged the surfaces by weight system. Each weight system is listed by the number assigned to it by Yonemura in [**32**]. In each weight system, we have listed all possible invertible polynomials of the form (1) with non-symplectic automorphism of order m, and for each polynomial, we list the orders of the possible groups G/J_W satisfying $J_W \subseteq G \subseteq SL_W$. The invariants $(r, q_{S_X(\sigma_m)})$ are then given, as well as the number of the BHK mirror dual. Finally, we have also indicated which method was used to determine $q_{S_X(\sigma_m)}$.

When consulting the tables, it will be helpful to know that $w_{5,1}^{\epsilon} = -w_{5,1}^{\epsilon}$, and that $4w_{3,1}^{-1} = 4w_{3,1}^1$, $4w_{2,1}^{-1} = 4w_{2,1}^1$. The first fact follows simply by definition. The latter two follow from [**27**, Theorem 1.8.2].

Table 2. Table for $m = 42$.

No.	Weights	Polynomial	SL/J	G/J	(r,q)	BHK dual	Method
14	(21,14,6,1;42)	$x^2 + y^3 + z^7 + w^{42}$	1	1	$(10, \langle 0 \rangle)$	14	I

Table 3. Table for $m = 30$.

No.	Weights	Polynomial	SL/J	G/J	(r,q)	BHK dual	Method
38	(15,8,6,1;30)	$x^2 + y^3 z + z^5 + w^{30}$	1	1	$(11, w_{2,1}^{-1})$	50	I
50	(15,10,4,1;30)	$x^2 + y^3 + yz^5 + w^{30}$	1	1	$(9, w_{2,1}^1)$	38	I

Table 4. Table for $m = 24$.

No.	Weights	Polynomial	SL/J	G/J	(r, q)	BHK dual	Method
13a	(12,8,3,1;24)	$x^2 + y^3 + xz^4 + w^{24}$	1	1	$(8, w_{3,1}^{-1})$	20	I
13b	(12,8,3,1;24)	$x^2 + y^3 + z^8 + w^{24}$	2	2	$(12, w_{3,1}^{1})$	13b	I
				1	$(8, w_{3,1}^{-1})$	13b	I
20	(9,8,6,1;24)	$x^2z + y^3 + z^4 + w^{24}$	1	1	$(12, w_{3,1}^{1})$	13a	I

Table 5. Table for $m = 22$.

No.	Weights	Polynomial	SL/J	G/J	(r, q)	BHK dual	Method
78	(11,6,4,1;22)	$x^2 + y^3z + yz^4 + w^{22}$	1	1	$(10, w_{2,1}^{-1} \oplus w_{2,1}^{1})$	78	II

Table 6. Table for $m = 20$.

No.	Weights	Polynomial	SL/J	G/J	(r, q)	BHK dual	Method
9a	(10,5,4,1;20)	$x^2 + xy^2 + z^5 + w^{20}$	1	1	$(10, w_{5,1}^{-1})$	9a	I
9b	(10,5,4,1;20)	$x^2 + y^4 + z^5 + w^{20}$	2	2	$(10, w_{5,1}^{-1})$	9b	I
				1	$(10, w_{5,1}^{-1})$	9b	I

Table 7. Table for $m = 18$.

No.	Weights	Polynomial	SL/J	G/J	(r, q)	BHK dual	Method
12a	(9,6,2,1;18)	$x^2 + y^3 + yz^6 + w^{18}$	2	2	$(11, w_{2,1}^{1} \oplus w_{3,1}^{1})$	39a	I
				1	$(6, v)$	39a	I
12b	(9,6,2,1;18)	$x^2 + y^3 + z^9 + w^{18}$	3	3	$(14, v)$	12b	I
				1	$(6, v)$	12b	I
39a	(9,5,3,1;18)	$x^2 + y^3z + z^6 + w^{18}$	2	2	$(14, v)$	12a	I
				1	$(9, w_{2,1}^{-1} \oplus w_{3,1}^{-1})$	12a	I
39b	(9,5,3,1;18)	$x^2 + y^3z + xz^3 + w^{18}$	1	1	$(9, w_{2,1}^{-1} \oplus w_{3,1}^{-1})$	60	I
60	(7,6,4,1;18)	$x^2z + y^3 + yz^3 + w^{18}$	1	1	$(11, w_{2,1}^{1} \oplus w_{3,1}^{1})$	39b	I

Table 8. Table for $m = 16$.

No.	Weights	Polynomial	SL/J	G/J	(r, q)	BHK dual	Method
37a	(8,4,3,1;16)	$x^2 + xy^2 + yz^4 + w^{16}$	1	1	$(9, w_{2,3}^{5})$	58	III
37b	(8,4,3,1;16)	$x^2 + y^4 + yz^4 + w^{16}$	2	2	$(11, w_{2,3}^{-5})$	37b	IV
				1	$(9, w_{2,3}^{5})$	37b	III
58	(6,5,4,1;16)	$x^2z + xy^2 + z^4 + w^{16}$	1	1	$(11, w_{2,3}^{-5})$	37a	IV

Table 9. Table for $m = 15$.

No.	Weights	Polynomial	SL/J	G/J	(r, q)	BHK dual	Method
11a	(15,10,3,2;30)	$x^2 + y^3 + xz^5 + w^{15}$	1	1	$(10, w_{3,1}^{-1} \oplus w_{3,1}^{1})$	22a	II
11b	(15,10,3,2;30)	$x^2 + y^3 + z^{10} + w^{15}$	1	1	$(10, w_{3,1}^{-1} \oplus w_{3,1}^{1})$	11b	II
22a	(6,5,3,1;15)	$x^2z + y^3 + z^5 + w^{15}$	1	1	$(10, w_{3,1}^{-1} \oplus w_{3,1}^{1})$	11a	II
22b	(6,5,3,1;15)	$x^2z + y^3 + xz^3 + w^{15}$	1	1	$(10, w_{3,1}^{-1} \oplus w_{3,1}^{1})$	22b	II

Table 10. Table for $m = 14$.

No.	Weights	Polynomial	SL/J	G/J	(r, q)	BHK dual	Method
40a	(7,4,2,1;14)	$x^2 + y^3z + z^7 + w^{14}$	1	1	$(7, v \oplus w_{2,1}^{-1})$	47	II
40b	(7,4,2,1;14)	$x^2 + y^3z + yz^5 + w^{14}$	2	2	$(13, v \oplus w_{2,1}^{1})$	40b	II
				1	$(7, v \oplus w_{2,1}^{-1})$	40b	II
47	(21,14,4,3;42)	$x^2 + y^3 + yz^7 + w^{14}$	1	1	$(13, v \oplus w_{2,1}^{1})$	40a	II

Table 11. Table for $m = 10$.

No.	Weights	Polynomial	SL/J	G/J	(r, q)	BHK dual	Method
6a	(5,2,2,1;10)	$x^2 + y^4z + z^5 + w^{10}$	2	2	$(8, w_{5,1}^{-1} \oplus 2w_{2,1}^{1})$	36a	I
				1	$(6, u \oplus v)$	36a	II
6b	(5,2,2,1;10)	$x^2 + y^5 + z^5 + w^{10}$	5	5	$(14, u \oplus v)$	6b	II
				1	$(6, u \oplus v)$	6b	II
6c	(5,2,2,1;10)	$x^2 + y^4z + yz^4 + w^{10}$	3	3	$(14, u \oplus v)$	6c	II
				1	$(6, u \oplus v)$	6c	II
11a	(15,10,3,2;30)	$x^2 + y^3 + z^{10} + yw^{10}$	2	2	$(17, w_{2,1}^{1})$	42a	I
				1	$(10, v \oplus v)$	42a	II
11b	(15,10,3,2;30)	$x^2 + y^3 + z^{10} + w^{15}$	1	1	$(10, v \oplus v)$	11b	II
36a	(10,5,3,2;20)	$x^2 + y^4 + yz^5 + w^{10}$	2	2	$(14, u \oplus v)$	6a	II
				1	$(12, w_{5,1}^{-1} \oplus 2w_{2,1}^{-1})$	6a	I
36b	(10,5,3,2;20)	$x^2 + xy^2 + yz^5 + w^{10}$	1	1	$(12, w_{5,1}^{-1} \oplus 2w_{2,1}^{-1})$	63	I
42a	(5,3,1,1;10)	$x^2 + y^3w + z^{10} + w^{10}$	2	2	$(10, v \oplus v)$	11a	II
				1	$(3, w_{2,1}^{-1})$	11a	I
42b	(5,3,1,1;10)	$x^2 + y^3z + xz^5 + w^{10}$	1	1	$(3, w_{2,1}^{-1})$	68	I
42c	(5,3,1,1;10)	$x^2 + y^3z + yz^7 + w^{10}$	4	4	$(17, w_{2,1}^{1})$	42c	I
				2	$(10, v \oplus v)$	42c	II
				1	$(3, w_{2,1}^{-1})$	42c	I
63	(4,3,2,1;10)	$x^2z + y^2x + z^5 + w^{10}$	1	1	$(8, w_{5,1}^{-1} \oplus 2w_{2,1}^{1})$	36b	I
68	(13,10,4,3;30)	$x^2z + y^3 + yz^5 + w^{10}$	1	1	$(17, w_{2,1}^{1})$	42b	I

Table 12. Table for $m = 9$.

No.	Weights	Polynomial	SL/J	G/J	(r, q)	BHK dual	Method
12a	(9,6,2,1;18)	$x^2 + y^3 + z^9 + xw^9$	3	3	$(16, w_{3,1}^{-1})$	25a	I
				1	$(4, w_{3,1}^{1})$	25a	I
12b	(9,6,2,1;18)	$x^2 + y^3 + z^9 + yw^{12}$	1	1	$(4, w_{3,1}^{1})$	43a	I
12c	(9,6,2,1;18)	$x^2 + y^3 + z^9 + w^{18}$	3	3	$(16, w_{3,1}^{-1})$	12c	I
				1	$(4, w_{3,1}^{1})$	12c	I
18a	(3,3,2,1;9)	$x^3 + y^3 + xz^3 + w^9$	3	3	$(12, w_{3,1}^{1} \oplus w_{3,2}^{1})$	18a	IV
				1	$(8, w_{3,1}^{-1} \oplus w_{3,2}^{-1})$	18a	III
18b	(3,3,2,1;9)	$x^3 + xy^2 + yz^3 + w^9$	2	2	$(12, w_{3,1}^{1} \oplus w_{3,2}^{1})$	18b	IV
				1	$(8, w_{3,1}^{-1} \oplus w_{3,2}^{-1})$	18b	III
25a	(4,3,1,1;9)	$x^2w + y^3 + z^9 + w^9$	3	3	$(16, w_{3,1}^{-1})$	12a	I
				1	$(4, w_{3,1}^{1})$	12a	I
25b	(4,3,1,1;9)	$x^2w + y^3 + z^9 + yw^6$	1	1	$(4, w_{3,1}^{1})$	43b	I
25c	(4,3,1,1;9)	$x^2w + y^3 + z^9 + xw^5$	3	3	$(16, w_{3,1}^{-1})$	25a	I
				1	$(4, w_{3,1}^{1})$	25a	I
43a	(18,11,4,3;36)	$x^2 + y^3w + z^9 + w^{12}$	1	1	$(16, w_{3,1}^{-1})$	12b	I
43b	(18,11,4,3;36)	$x^2 + y^3w + z^9 + xw^6$	1	1	$(16, w_{3,1}^{-1})$	25b	I

Table 13. Table for $m = 6$.

No.	Weights	Polynomial	SL/J	G/J	(r, q)	BHK dual	Method
2a	(4,3,3,2;12)	$x^3 + y^3z + z^4 + w^6$	3	3	$(16, v \oplus w_{3,1}^1)$	3a	I
				1	$(10, 4w_{3,1}^1)$	3a	II
2b	(4,3,3,2;12)	$x^3 + y^3z + yz^3 + w^6$	1	1	$(10, 4w_{3,1}^1)$	2b	II
2c	(4,3,3,2;12)	$x^3 + y^4 + z^4 + w^6$	2	2	$(10, 4w_{3,1}^1)$	2c	II
				1	$(10, 4w_{3,1}^1)$	2c	II
3a	(2,2,1,1;6)	$x^3 + y^3 + yz^4 + w^6$	3	3	$(10, 4w_{3,1}^1)$	2a	II
				1	$(4, v \oplus w_{3,1}^{-1})$	2a	I
3b	(2,2,1,1;6)	$x^2y + y^3 + z^6 + w^6$	6	6	$(19, w_{2,1}^{-1})$	5a	I
				3	$(16, v \oplus w_{3,1}^1)$	5a	I
				2	$(9, 3w_{2,1}^{-1} \oplus 2w_{3,1}^{-1})$	5a	I
				1	$(4, v \oplus w_{3,1}^{-1})$	5a	I
3c	(2,2,1,1;6)	$x^3 + xy^2 + yz^4 + w^6$	1	1	$(4, v \oplus w_{3,1}^{-1})$	57	I
3d	(2,2,1,1;6)	$x^3 + y^3 + z^6 + w^6$	9	9	$(16, v \oplus w_{3,1}^1)$	3d	I
				3	$(16, v \oplus w_{3,1}^1)$	3d	I
				3	$(10, 4w_{3,1}^1)$	3d	II
				3	$(10, 4w_{3,1}^1)$	3d	II
				3	$(4, v \oplus w_{3,1}^{-1})$	3d	I
				1	$(4, v \oplus w_{3,1}^{-1})$	3d	I
3e	(2,2,1,1;6)	$x^2y + xy^2 + z^6 + w^6$	3	3	$(16, v \oplus w_{3,1}^1)$	3e	I
				1	$(4, v \oplus w_{3,1}^{-1})$	3e	I
5a	(3,1,1,1;6)	$x^2 + xy^3 + z^6 + w^6$	6	6	$(16, v \oplus w_{3,1}^1)$	3b	I
				3	$(11, 3w_{2,1}^1 \oplus 2w_{3,1}^1)$	3b	I
				2	$(4, v \oplus w_{3,1}^{-1})$	3b	I
				1	$(1, w_{2,1}^1)$	3b	I
5b	(3,1,1,1;6)	$x^2 + y^5w + z^6 + w^6$	2	2	$(8, 6w_{2,1}^{-1})$	29	II
				1	$(1, w_{2,1}^1)$	29	I
5c	(3,1,1,1;6)	$x^2 + xy^3 + yz^5 + w^6$	1	1	$(1, w_{2,1}^1)$	56	I
5d	(3,1,1,1;6)	$x^2 + y^6 + z^5w + zw^5$	8	8	$(19, w_{2,1}^{-1})$	5d	I
				4	$(12, 6w_{2,1}^1)$	5d	II
				2	$(8, 6w_{2,1}^{-1})$	5d	II
				1	$(1, w_{2,1}^1)$	5d	I
5e	(3,1,1,1;6)	$x^2 + y^6 + z^6 + w^6$	12	12	$(19, w_{2,1}^{-1})$	5e	I
				6	$(12, 6w_{2,1}^1)$	5e	II
				4	$(9, 3w_{2,1}^{-1} \oplus 2w_{3,1}^{-1})$	5e	I
				3	$(11, 3w_{2,1}^1 \oplus 2w_{3,1}^1)$	5e	I
				2	$(8, 6w_{2,1}^{-1})$	5e	II
				1	$(1, w_{2,1}^1)$	5e	I
8a	(6,3,2,1;12)	$x^2 + y^4 + z^6 + xw^6$	2	2	$(14, 2w_{2,1}^{-1} \oplus w_{3,1}^1)$	23	I
				1	$(6, 2w_{2,1}^1 \oplus w_{3,1}^{-1})$	23	I
8b	(6,3,2,1;12)	$x^2 + y^4 + z^6 + yw^9$	2	2	$(10, v \oplus 4w_{2,1}^{-1})$	33a	II
				1	$(6, 2w_{2,1}^1 \oplus w_{3,1}^{-1})$	33a	I
8c	(6,3,2,1;12)	$x^2 + xy^2 + z^6 + yw^9$	1	1	$(6, 2w_{2,1}^1 \oplus w_{3,1}^{-1})$	70	I
8d	(6,3,2,1;12)	$x^2 + y^4 + z^6 + w^{12}$	4	4	$(14, 2w_{2,1}^{-1} \oplus w_{3,1}^1)$	8d	I
				2	$(14, 2w_{2,1}^{-1} \oplus w_{3,1}^1)$	8d	I
				2	$(6, 2w_{2,1}^1 \oplus w_{3,1}^{-1})$	8d	I
				2	$(10, v \oplus 4w_{2,1}^{-1})$	8d	II
				1	$(6, 2w_{2,1}^1 \oplus w_{3,1}^{-1})$	8d	I
8e	(6,3,2,1;12)	$x^2 + xy^2 + z^6 + w^{12}$	2	2	$(14, 2w_{2,1}^{-1} \oplus w_{3,1}^1)$	8e	I
				1	$(6, 2w_{2,1}^1 \oplus w_{3,1}^{-1})$	8e	I

No.	Weights	Polynomial	SL/J	G/J	(r,q)	BHK dual	Method
23	(5,3,2,2;12)	$x^2w+y^4+z^6+w^6$	2	2	$(14, 2w_{2,1}^{-1}\oplus w_{3,1}^{1})$	8a	I
				1	$(6, 2w_{2,1}^{1}\oplus w_{3,1}^{-1})$	8a	I
29	(15,6,5,4;30)	$x^2+y^5+z^6+yw^6$	2	2	$(19, w_{2,1}^{-1})$	5b	I
				1	$(12, 6w_{2,1}^{1})$	5b	II
33a	(9,4,3,2;18)	$x^2+y^4w+z^6+w^9$	2	2	$(14, 2w_{2,1}^{-1}\oplus w_{3,1}^{1})$	8b	I
				1	$(10, v\oplus 4w_{2,1}^{-1})$	8b	II
33b	(9,4,3,2;18)	$x^2+y^4w+z^6+yw^7$	1	1	$(10, v\oplus 4w_{2,1}^{-1})$	33b	II
56	(11,8,6,5;30)	$x^2y+y^3z+z^5+w^6$	1	1	$(19, w_{2,1}^{-1})$	5c	I
57	(9,6,5,4;24)	$x^2y+y^4+xz^3+w^6$	1	1	$(16, v\oplus w_{3,1}^{1})$	3c	I
70	(8,5,3,2;18)	$x^2w+xy^2+z^6+w^9$	1	1	$(14, 2w_{2,1}^{-1}\oplus w_{3,1}^{1})$	8c	I

A. Computer code for computing lattices.

In order to compute the lattices using the configuration of curves, we used the following Magma code, developed by Antonio Laface and added here with his permission.

This first function takes an even bilinear form B, and outputs generators of the discriminant group and the values of q_B on these generators.

```
disc:=function(M)
 S,A,B:=SmithForm(M);
 l:=[[S[i,i],i]: i in [1..NumberOfColumns(S)]| S[i,i] notin {0,1}];
 sA:=Matrix(Rationals(),ColumnSubmatrixRange(B,l[1][2],l[#l][2]));
 for i in [1..#l] do
  MultiplyColumn(~sA,1/l[i][1],i);
 end for;
 Q:=Transpose(sA)*Matrix(Rationals(),M)*sA;
 for i,j in [1..NumberOfColumns(Q)] do
  if i ne j then
   Q[i,j]:=Q[i,j]-Floor(Q[i,j]);
  else
   Q[i,j]:=Q[i,j]-Floor(Q[i,j])+ (Floor(Q[i,j]) mod 2);
  end if;
 end for;
 return [l[i][1]: i in [1..#l]], Q;
end function;
```

The next function determines whether a given even bilinear form has overlattices. Input is a matrix M and a number n. The output is the subgroup of A_L that takes values equal to n modulo $2\mathbb{Z}$. For isotropic subgroups of the discriminant group, use $n=0$.

```
isot:=function(M,n)
 v,U:=disc(M);
 Q:=Rationals();
 A:=AbelianGroup(v);
 return [Eltseq(a) : a in A |
 mod2(Matrix(Q,1,#v,Eltseq(a))*U*Matrix(Q,#v,1,Eltseq(a)))[1,1] eq n];
end function;
```

The function `mod2` is as follows:

```
mod2:=function(Q);
 for i,j in [1..Nrows(Q)] do
  if i ne j then Q[i,j]:=Q[i,j]-Floor(Q[i,j]);
  else Q[i,j]:=Q[i,j]-2*Floor(Q[i,j]/2);
  end if;
 end for;
 return Q;
end function;
```

Finally, the following function compares two discriminant quadratic forms, and lets us know if they are the same finite quadratic form or not. This is not always easy to check due to the relations in Proposition 1.2.

```
dicompare:=function(M,Q)
 v,U:=disc(M);
 w,D:=disc(Q);
 if v ne w then return false; end if;
 A:=AbelianGroup(v);
 Aut:=AutomorphismGroup(A);
 f,G:=PermutationRepresentation(Aut);
 h:=Inverse(f);
 ll:=[Matrix(Rationals(),[Eltseq(Image(h(g),A.i)) :
      i in [1..Ngens(A)]]) : g in G];
 dd:=[mod2(a*U*Transpose(a)) : a in ll];
 return D in dd;
end function;
```

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