Journal of Pure and Applied Algebra 223 (2019) 2123–2144

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

On the one-dimensional family of Riemann surfaces of genus q with $4q$ automorphisms ☆

Sebastián Reyes-Carocca

Departamento de Matemática y Estadística, Universidad de La Frontera, Avenida Francisco Salazar 01145, Temuco, Chile

ARTICLE INFO

Article history:
Received 27 February 2018
Received in revised form 27 June 2018
Available online 5 July 2018
Communicated by S. Kovács

MSC:
30F10; 14H37; 14H40

ABSTRACT

Bujalance, Costa and Izquierdo have recently proved that all those Riemann surfaces of genus $g \geq 2$ different from $3, 6, 12, 15$ and 30, with exactly $4g$ automorphisms form an equisymmetric one-dimensional family, denoted by $\mathcal{F}_g$. In this paper, for every prime number $q \geq 5$, we explore further properties of each Riemann surface S in $\mathcal{F}_q$ as well as of its Jacobian variety JS.

1. Introduction

Automorphism groups of compact Riemann surfaces have been extensively studied, going back to Wiman, Klein and Hurwitz, among others.

It is classically known that the full automorphism group of a Riemann surface of genus $g \geq 2$ is finite; its size is bounded by $84(g-1)$. Moreover, there are infinitely many integers g for which this bound is attained; see [36].

Usually when additional conditions are imposed on a group of automorphisms, a smaller bound for its order is obtained; for example, classical results assert that in the abelian and cyclic case these bounds are $4g+4$ and $4g+2$ respectively.

It is an interesting problem to understand the extent to which the order of the full automorphism group determines the Riemann surface; see for example [32], [33] and [39].

Very recently, Costa and Izquierdo have proved that the maximal order of the form $ag+b$ (for fixed integers a and b) of the full automorphism group of equisymmetric and one-dimensional families of Riemann surfaces of genus $g \geq 2$ appearing in all genera is $4g+4$. Moreover, they constructed explicit families attaining this bound; see [11]. The second possible largest order is $4g$.

☆ Partially supported by Postdoctoral FONDECYT Grant 3160002, Redes Etapa Inicial Grant REDI-170071 and Anillo ACT1415 PIA CONICYT Grant.

E-mail address: sebastian.reyes@ufrontera.cl.

https://doi.org/10.1016/j.jpaa.2018.07.011
0022-4049/

Bujalance, Costa and Izquierdo have recently determined all those Riemann surfaces of genus $g \geq 2$ with exactly $4g$ automorphisms. More precisely, following [6, Theorem 7], if g is different from the exceptional values $3, 6, 12, 15$ and 30, then the Riemann surfaces of genus g admitting exactly $4g$ automorphisms form an equisymmetric one-dimensional family, denoted by $\mathcal{F}_g$. Moreover, if S is a Riemann surface in $\mathcal{F}_g$ then its full automorphism group G is isomorphic to a dihedral group, and the corresponding quotient S/G has genus zero.

The present article is devoted to study further properties of each member S of the family $\mathcal{F}_g$ and of its Jacobian variety JS. In spite of the fact that the results of this paper might be stated for each integer $g \geq 2$ different from $3, 6, 12, 15$ and 30, for the sake of simplicity we shall restrict to the case $g = q \geq 5$ prime.

This paper is organized as follows.

In Section 2 we shall introduce the basic background; namely, group actions on Riemann surfaces, complex tori and abelian varieties, representation of groups, and the group algebra decomposition theorem for Jacobians.

In Section 3 we shall take advantage of the hyperellipticity of the Riemann surfaces in the family $\mathcal{F}_q$ (see [6, Remark 9]) to determine explicit algebraic descriptions of them. In addition, with respect to these models, we will provide realizations of their full automorphism groups.

If a finite group G acts on a Riemann surface S, then this action induces a G-equivariant isogeny decomposition of JS into a product of abelian subvarieties, of the form $JS \sim_G \Pi_{i=1}^{r} B_i^{n_i}$. This decomposition (known as the *group algebra decomposition* of JS with respect to G; see [8] and [34]) only depends on the algebraic structure of the group; however, further information such as the dimension of each factor B_i depends on the geometry of the action. Following [44], the dimension of each B_i is explicitly given after choosing a *generating vector* of G representing the action on S. Because of this dependence, the general question arises about how the choice of such a generating vector affects the group algebra decomposition of JS.

In Section 4 we shall give a complete answer to the aforementioned question, for each Riemann surface S in the family $\mathcal{F}_q$. To prove this result, we shall begin by proving some lemmata concerning the rational representations of G, and the possible generating vectors representing the action. We shall also prove that each Jacobian JS contains an elliptic curve, and that it decomposes into a product of Jacobians of quotients of S.

In Section 5 we shall explore the fields of definition of the Riemann surfaces S in the family $\mathcal{F}_q$. More precisely, we shall give a characterization for when S and JS can be defined, as projective algebraic varieties, by polynomials with real coefficients, and by polynomials with algebraic coefficients. Furthermore, in the latter case we prove that JS decomposes in terms of abelian subvarieties which can also be defined by polynomials with algebraic coefficients. We shall also observe that S and JS can be defined over the field of moduli of S.

Finally, in Section 6 we shall compute the dimension of a *special variety* in the moduli space $\mathcal{A}_q$ of principally polarized abelian varieties of dimension q associated to each Riemann surface S in the family $\mathcal{F}_q$, called the *Shimura family*; see [54]. Moreover, for the particular case $q = 5$, we will be able to describe its elements – by exhibiting period matrices – as members of a three-dimensional family in $\mathcal{A}_5$ admitting a fixed action of the dihedral group of order 20.

Acknowledgments. The author wishes to express his gratitude to Anita M. Rojas for sharing her SAGE routines with him.

2. Preliminaries

2.1. Group actions on Riemann surfaces

Let S be a compact Riemann surface and let G be a finite group. We denote by $\mathrm{Aut}(S)$ the full automorphism group of S, and say that G acts on S if there is a group monomorphism $\psi : G \to \mathrm{Aut}(S)$. The space of orbits S/G of the action of $G \cong \psi(G)$ on S is naturally endowed with a Riemann surface structure in such a way that the natural projection $\pi : S \to S/G$ is holomorphic. The degree of π is the order $|G|$ of G and the multiplicity of π at $p \in S$ is $|G_p|$, where G_p denotes the stabilizer of p in G. If $|G_p| \neq 1$ then p is called a *branch point* and its image by π is a *branch value*.

Let $\{p_1, \dots, p_l\}$ be a maximal collection of non-G-equivalent branch points of π. The *signature* of the action of G on S is the tuple $(\gamma; m_1, \dots, m_l)$ where γ is the genus of the quotient S/G and $m_i = |G_{p_i}|$. If $\gamma = 0$ we write $(m_1, \dots, m_r)$ for short. The branch value $\pi(p_i)$ is said to be *marked* with m_i. The Riemann–Hurwitz formula relates these numbers, the order of G and the genus g of S; namely

$$2g - 2 = |G|(2\gamma - 2) + |G| \cdot \Sigma_{i=1}^{l}(1 - \tfrac{1}{m_i}).$$

A $2\gamma + l$ tuple $(a_1, \dots, a_\gamma, b_1, \dots, b_\gamma, c_1, \dots, c_l)$ of elements of G is called a *generating vector of G of type* $(\gamma; m_1, \dots, m_l)$ if the following conditions are satisfied:

(a) G is generated by the elements $a_1, \dots, a_\gamma, b_1, \dots, b_\gamma, c_1, \dots, c_l$,
(b) order$(c_i) = m_i$ for $1 \leq i \leq l$, and
(c) $\Pi_{j=1}^{\gamma}[a_j, b_j]\Pi_{i=1}^{l}c_i = 1$, where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$.

Riemann's existence theorem ensures that the group G acts on a Riemann surface of genus g with signature $(\gamma; m_1, \dots, m_l)$ if and only if the Riemann–Hurwitz formula is satisfied and G has a generating vector of type $(\gamma; m_1, \dots, m_l)$. See [4].

If we denote by C_j the conjugacy class of the subgroup G_{p_j} in G then, the tuple $(\gamma; [m_1, C_1], \dots, [m_l, C_l])$ is called a *geometric signature* for the action of G on S. This concept was introduced in [44] in order to control the behavior of the intermediate coverings ($S \to S/H$ for a subgroup H of G) and the dimension of the factors arising in the group algebra decomposition of JS (see Subsection 2.6).

We shall say that the geometric signature $(\gamma; [m_1, C_1], \dots, [m_l, C_l])$ is *associated* to the generating vector $(a_1, \dots, a_\gamma, b_1, \dots, b_\gamma, c_1, \dots, c_l)$ because the subgroup of G generated by c_i is in the conjugacy class C_i.

2.2. Topologically equivalent actions

Let $\mathrm{Hom}^+(S)$ denote the group of orientation preserving homeomorphisms of S. Two actions ψ_1 and ψ_2 of G on S are *topologically equivalent* if there exist $\omega \in \mathrm{Aut}(G)$ and $h \in \mathrm{Hom}^+(S)$ such that

$$\psi_2(g) = h\psi_1(\omega(g))h^{-1} \quad \text{for all } g \in G. \tag{2.1}$$

In terms of Fuchsian groups, the action of G on S can be constructed by means of a pair of Fuchsian groups $\Gamma \trianglelefteq \Delta$ such that $S = \mathbb{H}/\Gamma$, with $\mathbb{H}$ denoting the upper half-plane, and an epimorphism $\theta : \Delta \to G$ with kernel Γ. The group Γ is torsion-free and isomorphic to the fundamental group of S. It is also known that Δ has a presentation given by generators $\alpha_1, \dots, \alpha_\gamma, \beta_1, \dots, \beta_\gamma, \gamma_1, \dots, \gamma_l$ and relations

$$\gamma_1^{m_1} = \dots = \gamma_l^{m_l} = \Pi_{j=1}^{\gamma}[\alpha_j, \beta_j]\Pi_{i=1}^{l}\gamma_i = 1.$$

Note that there is a bijective correspondence between the set of generating vectors of G of type $(\gamma, m_1, \dots, m_l)$ and the set $\mathscr{K}$ of epimorphism of groups $\Delta \to G$ with torsion-free kernel.

Each orientation preserving homeomorphism h satisfying (2.1) yields a group automorphism h^* of Δ; we denote by $\mathscr{B}$ the subgroup of $\mathrm{Aut}(\Delta)$ consisting of them. The group $\mathrm{Aut}(G) \times \mathscr{B}$ acts on $\mathscr{K}$ by

$$((\omega, h^*), \theta) \mapsto \omega \circ \theta \circ (h^*)^{-1}$$

and therefore it also acts on the set of generating vectors of G of type $(\gamma, m_1, \dots, m_l)$.

Let σ_1 and σ_2 be two generating vectors of type $(\gamma, m_1, \dots, m_l)$ of G. Then σ_1 and σ_2 define topologically equivalent actions if and only if σ_1 and σ_2 are in the same $(\mathrm{Aut}(G) \times \mathscr{B})$-orbit (see [4]; also [22] and [35]).

We refer to the classical articles [48] and [49] for more details concerning the relationship between Riemann surfaces, generating vectors and Fuchsian groups.

2.3. Abelian varieties

A g-dimensional *complex torus* $X = V/\Lambda$ is the quotient of a g-dimensional complex vector space V by a lattice Λ. Each complex torus is an abelian group and a g-dimensional compact connected complex analytic manifold.

Complex tori can be described in a very concrete way, as follows. Choose bases

$$B_V = \{v_i\}_{i=1}^{g} \quad \text{and} \quad B_\Lambda = \{\lambda_j\}_{j=1}^{2g} \tag{2.2}$$

of V as a $\mathbb{C}$-vector space, and of Λ as a $\mathbb{Z}$-module, respectively. Then there are complex constants $\{\pi_{ij}\}$ such that $\lambda_j = \Sigma_{i=1}^{g} \pi_{ij} v_i$. The matrix

$$\Pi = (\pi_{ij}) \in \mathrm{M}(g \times 2g, \mathbb{C})$$

represents X, and is known as the *period matrix* for X with respect to (2.2).

A *homomorphism* between complex tori is a holomorphic map which is also a homomorphism of groups. We shall denote by $\mathrm{End}(X)$ the ring of endomorphism of X. An *automorphism* of a complex torus is a bijective homomorphism into itself.

Special homomorphisms are *isogenies:* these are surjective homomorphisms with finite kernel; isogenous tori are denoted by $X_1 \sim X_2$. The isogenies of a complex torus X into itself are the invertible elements of the ring of rational endomorphisms

$$\mathrm{End}_{\mathbb{Q}}(X) := \mathrm{End(X)} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

An *abelian variety* is a complex torus which is also a projective algebraic variety. Each abelian variety $X = V/\Lambda$ admits a *polarization*; namely, a non-degenerate real alternating form Θ on V such that for all $v, w \in V$

$$\Theta(iv, iw) = \Theta(v, w) \quad \text{and} \quad \Theta(\Lambda \times \Lambda) \subset \mathbb{Z}.$$

If the elementary divisors of $\Theta|_{\Lambda\times\Lambda}$ are $\{1, \overset{g}{\dots}, 1\}$, where g is the dimension of X, then the polarization Θ is called *principal* and the pair (X, Θ) is called a *principally polarized abelian variety* (from now on we write *ppav* for short).

Let $(X = V/\Lambda, \Theta)$ be a ppav of dimension g. Then there exists a basis for Λ such that the matrix for Θ with respect to it is

$$J = \left(\begin{smallmatrix} 0 & I_g \\ -I_g & 0 \end{smallmatrix}\right)$$

with I_g denoting the $g \times g$ identity matrix; such a basis is termed *symplectic.* Furthermore, there exist a basis for V and a symplectic basis for Λ which respect to which the period matrix for X is $\Pi = (I_g\, Z)$, where Z belongs to the Siegel space

$$\mathscr{H}_g = \{Z \in \mathrm{M}(g, \mathbb{C}) : Z = Z^t \,\text{and}\, \mathrm{Im}(Z) > 0\},$$

with Z^t denoting the transpose matrix of Z.

2.4. Moduli space of ppavs

An *isomorphism* between ppavs is an isomorphism of the underlying complex torus structures that preserves the polarizations.

Let (X_i, Θ_i) be a ppav of dimension g, and let $\Pi_i = (I_g \, Z_i)$ be the period matrix of X_i with respect to chosen basis, for $i = 1, 2$. Each isomorphism between (X_1, Θ_1) and (X_2, Θ_2) is given by a pair of matrices

$$M \in \mathrm{GL}(g, \mathbb{C}) \quad \text{and} \quad R \in \mathrm{GL}(2g, \mathbb{Z})$$

(corresponding to the *analytic* and *rational* representations, respectively) such that

$$M(I_g \, Z_1) = (I_g \, Z_2)R. \tag{2.3}$$

Since R preserves the principal polarizations, it belongs to the symplectic group

$$\mathrm{Sp}(2g, \mathbb{Z}) = \{R \in \mathrm{M}(2g, \mathbb{Z}) : R^t J R = J\}.$$

Now, it follows from (2.3) that the correspondence

$$\mathrm{Sp}(2g, \mathbb{Z}) \times \mathscr{H}_g \to \mathscr{H}_g \qquad \left(\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), Z\right) \mapsto (A + ZC)^{-1}(B + ZD)$$

defines an action which identifies period matrices representing isomorphic ppavs. Hence, the quotient

$$\mathscr{H}_g \to \mathcal{A}_g := \mathscr{H}_g / \mathrm{Sp}(2g, \mathbb{Z})$$

is the *moduli space* of isomorphism classes of ppavs of dimension g.

We refer to [3] and [12] for more details on abelian varieties.

2.5. Representations of groups

Let G be a finite group and let $\rho : G \to \mathrm{GL}(V)$ be a complex representation of G. Abusing notation, we shall also write V to refer to the representation ρ. The *degree* d_V of V is the dimension of V as complex vector space, and the character χ_V of V is the map obtained by associating to each $g \in G$ the trace of the matrix $\rho(g)$. Two representations V_1 and V_2 are *equivalent* if and only if their characters agree; we write $V_1 \cong V_2$. The *character field* K_V of V is the field obtained by extending the rational numbers by the values of the character χ_V. The *Schur index* s_V of V is the smallest positive integer such that there exists a field extension L_V of K_V of degree s_V over which V can be defined.

It is a known fact that for each rational irreducible representation W of G there is a complex irreducible representation V of G such that

$$W \otimes_{\mathbb{Q}} \mathbb{C} \cong (\oplus_\sigma V^\sigma) \oplus \overset{s_V}{\cdots} \oplus (\oplus_\sigma V^\sigma) = s_V \, (\oplus_\sigma V^\sigma),$$

where the sum $\oplus_\sigma$ is taken over the Galois group associated to the extension $\mathbb{Q} \leq K_V$. We say that V is *associated* to W.

We refer to [47] for further basic facts related to representations of groups.

2.6. Group algebra decomposition theorem for Jacobians

Let S be a Riemann surface of genus g. Let us denote by $\mathscr{H}^{1,0}(S,\mathbb{C})$ the g-dimensional complex vector space of the holomorphic 1-forms on S, and by $H_1(S,\mathbb{Z})$ the first homology group of S. The *Jacobian variety* of S is the ppav of dimension g defined as

$$JS = (\mathscr{H}^{1,0}(S,\mathbb{C}))^*/H_1(S,\mathbb{Z}),$$

endowed with the principal polarization given by the geometric intersection number.

The relevance of the Jacobian variety lies in the well-known Torelli's theorem, which asserts that two compact Riemann surfaces are isomorphic if and only if their Jacobians are isomorphic as ppavs.

Let G be a finite group and let $W_1,\dots,W_r$ be its rational irreducible representations. It is known that each action of G on S induces a $\mathbb{Q}$-algebra homomorphism $\Phi : \mathbb{Q}[G] \to \mathrm{End}_{\mathbb{Q}}(JS)$. Each $\alpha \in \mathbb{Q}[G]$ defines an abelian subvariety of JS; namely,

$$A_\alpha := \mathrm{Im}(\alpha) = \Phi(l\alpha)(JS) \subset JS,$$

where l is some positive integer chosen such that $l\alpha \in \mathrm{End}(JS)$.

The decomposition $1 = e_1 + \cdots + e_r \in \mathbb{Q}[G]$, where e_i is a central idempotent (uniquely determined and canonically computed from W_i) yields an isogeny

$$JS \sim A_{e_1} \times \cdots \times A_{e_r}$$

which is G-equivariant; this is called the *isotypical decomposition* of JS. See [34].

Additionally, there are idempotents $f_{i1},\dots,f_{in_i}$ such that $e_i = f_{i1} + \cdots + f_{in_i}$ where $n_i = d_{V_i}/s_{V_i}$, with V_i being a complex irreducible representation associated to W_i. These idempotents provide n_i subvarieties of JS which are isogenous to each other; let B_i be one of them, for each i. Then

$$JS \sim_G B_1^{n_1} \times \cdots \times B_r^{n_r} \tag{2.4}$$

which is called the *group algebra decomposition* of JS with respect to G. See [8].

If W_1 denotes the trivial representation, then $n_1 = 1$ and $B_1 \sim J(S/G)$.

Let H be a subgroup of G. It was proved in [8] that the group algebra decomposition (2.4) of JS with respect to G yields the following isogeny decomposition:

$$J(S/H) \sim B_1^{n_1^H} \times \cdots \times B_r^{n_r^H} \quad \text{where} \;\; n_i^H = d_{V_i}^H/s_{V_i} \tag{2.5}$$

with $d_{V_i}^H$ denoting the dimension of the vector subspace V_i^H of V_i consisting of those elements which are fixed under H.

The previous result provides a criterion to identify if a factor in (2.4) is isogenous to the Jacobian of a quotient of S (cf. [30]). Namely, if a subgroup N of G satisfies $d_{V_i}^N = s_{V_i}$ for some fixed $2 \le i \le r$ and $d_{V_l}^N = 0$ for all $l \neq i$ such that $B_l \neq 0$, then

$$B_i \sim J(S/N). \tag{2.6}$$

Let us now suppose that $\tau = (\gamma; [m_1, C_1], \dots, [m_l, C_l])$ is the geometric signature of the action of G on S. Let G_k be a representative of the conjugacy class C_k for $1 \le k \le l$. In [44] it was proved that the dimension of the factor B_i in (2.4) is

$$\dim(B_i) = k_i(d_{V_i}(\gamma - 1) + \tfrac{1}{2}\Sigma_{k=1}^{l}(d_{V_i} - d_{V_i}^{G_k})) \tag{2.7}$$

where k_i is the degree of the extension $\mathbb{Q} \leq L_{V_i}$. To avoid confusion, we shall write $\dim_\tau(B_i)$ instead of $\dim(B_i)$ to refer to the dependence on τ.

For decompositions of Jacobians with respect to special groups, see, for example, the articles [7], [9], [24], [29], [31], [40], [42], [43] and [46].

3. Algebraic description of $\mathcal{F}_q$

Let $q \geq 5$ be a prime number. Let S denote a Riemann surface in the family $\mathcal{F}_q$ and let

$$G = \langle r, s : r^{2q} = s^2 = (sr)^2 = 1 \rangle \cong \mathbf{D}_{2q}$$

denote its full automorphism group. We recall that the quotient Riemann surface S/G has genus zero, and that the associated $4q$-fold branched regular covering map

$$\pi_G : S \to S/G \cong \mathbb{P}^1$$

ramifies over four values; three ramification values marked with 2 and one ramification value marked with $2q$. We can assume the action to be represented by the generating vector $(s, sr^{-2}, r^q, r^{q+2})$. In addition, up to a Möbius transformation, we can assume the branch values to be $\infty, 0, 1$ marked with 2 and $\lambda \in \mathbb{C} - \{0, 1\}$ marked with $2q$.

As we will discuss later (see Remark 5), λ must be different from the exceptional values $-1, \frac{1}{2}, 2, \gamma, \gamma^2$ where $\gamma^3 = -1$. If we denote by

$$\Omega := \mathbb{C} - \{0, \pm 1, \tfrac{1}{2}, 2, \gamma, \gamma^2\}$$

the set of *admissible* parameters, then the family $\mathcal{F}_q$ can be understood by means of an everywhere maximal rank holomorphic map

$$h : \mathcal{F}_q \to \Omega$$

in such a way that the fibers of h agree with the Riemann surfaces in $\mathcal{F}_q$. See, for example, Section 6.2 in [27].

We denote by S_λ the Riemann surface $h^{-1}(\lambda)$ and by $G_\lambda \cong G$ its full automorphism group.

Theorem 1. *Let $\lambda \in \Omega$. Then S_λ is isomorphic to the Riemann surface defined by the normalization of the hyperelliptic algebraic curve*

$$y^2 = x(x^{2q} + 2\tfrac{1+\lambda}{1-\lambda}x^q + 1).$$

Proof. Following [6, Remark 9], the Riemann surface S_λ is hyperelliptic; the hyperelliptic involution being represented by r^q. In other words, the Riemann surface $R_\lambda := S_\lambda/\langle r^q \rangle$ has genus zero, and the associated two-fold branched regular covering map

$$\pi_1 : S_\lambda \to R_\lambda$$

ramifies over $2q + 2$ values; let us denote these values by $\alpha_1, \ldots, \alpha_{2q+2}$. Let

$$\pi_2 : R_\lambda \to R_\lambda/K \cong S_\lambda/G \cong \mathbb{P}^1$$

denote the $2q$-fold branched regular covering map associated to the action of the quotient group $K = G/\langle r^q \rangle \cong \mathbf{D}_q$ on R_λ. The following diagram commutes

$$\begin{array}{ccc} & S_\lambda & \\ {}^{\pi_G}\swarrow & & \searrow^{\pi_1} \\ \mathbb{P}^1 & \xleftarrow{\;\pi_2\;} & R_\lambda \end{array}$$

Claim. The ramification values of π_2 are: ∞ marked with two, 0 marked with two, and λ marked with q. Moreover, among the ramification values of π_1 only two of them are ramification points of π_2, these points lying over λ by π_2.

We proceed to study carefully the ramification data associated to the coverings in the previous commutative diagram. To do that, we follow [44, Section 3.1].

(a) The fiber over ∞ by π_G consists of $2q$ different points, say $\beta'_1, \dots, \beta'_{2q}$. The stabilizer subgroup of β'_j in G is of the form $\langle sr^{-2t} \rangle$ for a suitable choice of t. Now, as $|\langle sr^{-2t} \rangle \cap \langle r^q \rangle| = 1$ for each choice of t, it follows that β'_j is not a branch point of π_1. Thus, over ∞ by π_2 there are exactly q different points, say

$$\{\beta_1, \dots, \beta_q\} = \pi_1(\{\beta'_1, \dots, \beta'_{2q}\}),$$

showing that ∞ is a ramification value of π_2 marked with two.

(b) The fiber over 0 by π_G consists of $2q$ different points, say $\gamma'_1, \dots, \gamma'_{2q}$. As argued in (a), γ'_j is not a branch point of π_1 and over 0 by π_2 there are exactly q different points, say

$$\{\gamma_1, \dots, \gamma_q\} = \pi_1(\{\gamma'_1, \dots, \gamma'_{2q}\}),$$

showing that 0 is a ramification value of π_2 marked with two.

(c) The fiber over 1 by π_G consists of $2q$ different points, say $\alpha'_1, \dots, \alpha'_{2q}$. The stabilizer subgroup of α'_j in G is $\langle r^q \rangle$ and therefore α'_j is a branch point marked with two of π_1 for each j. Thus, over 1 by π_2 there are exactly $2q$ different points, say

$$\alpha_j = \pi_1(\alpha'_j), \quad \text{for } j \in \{1, \dots, 2q\}$$

showing that 1 is not ramification value of π_2.

(d) The fiber over λ by π_G consists of 2 different points, say α'_{2q+1} and α'_{2q+2}. The stabilizer subgroup of α'_{2q+1} and α'_{2q+2} in G is $\langle r \rangle$. Now, as $|\langle r \rangle \cap \langle r^q \rangle| = 2$, it follows that α'_{2q+1} and α'_{2q+2} are branch points of π_1 marked with 2. Thus, over λ by π_2 there are exactly two different points:

$$\alpha_{2q+1} = \pi_1(\alpha'_{2q+1}) \quad \text{and} \quad \alpha_{2q+2} = \pi_1(\alpha'_{2q+2}),$$

showing that λ is a ramification value of π_2 marked with q.

The proof of the claim is done.

Then, without loss of generality, we can suppose K to be generated by

$$a(z) = \omega_{2q}^2 z \quad \text{and} \quad b(z) = \tfrac{1}{z}$$

where $\omega_t = \exp(\frac{2\pi i}{t})$, and that:

(a) the q branch points of π_2 over ∞ are $\beta_j = \omega_{2q}^{2j-1}$ for $1 \leq j \leq q$.
(b) the q branch points of π_2 over 0 are $\gamma_j = \omega_{2q}^{2j}$ for $0 \leq j < q$.
(c) the two branch points of π_2 over λ are $\alpha_{2q+1} = 0$ and $\alpha_{2q+2} = \infty$.

It follows that S_λ is isomorphic to the Riemann surface defined by the normalization of the hyperelliptic algebraic curve

$$y^2 = x(x-\alpha_1)\cdots(x-\alpha_{2q}),$$

and it only remains to prove that $\alpha_1, \dots, \alpha_{2q}$ are the $2q$ different solutions of the polynomial equation

$$z^{2q} + 2\tfrac{1+\lambda}{1-\lambda}z^q + 1 = 0.$$

Now, by virtue of the claim, to accomplish this task we need to exhibit a $2q$-fold branched regular covering map $\Pi : \mathbb{P}^1 \to \mathbb{P}^1$ admitting $\langle a, b\rangle \cong \mathbf{D}_q$ as its deck group, such that $\Pi(\infty) = \Pi(0) = \lambda$ and

$$\Pi(\omega_{2q}^k) = \begin{cases} \infty & \text{if } k \text{ is odd;} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

It is straightforward to check that $\Pi(z) = \lambda \cdot \frac{z^{2q}-2z^q+1}{z^{2q}+2z^q+1}$ is the desired map, and the proof follows directly after noticing that

$$\{\alpha_1, \dots, \alpha_{2q}\} = \Pi^{-1}(1). \quad \square$$

Remark 1. As we shall see later (see Theorem 5) among the members of the family $\mathcal{F}_q$ there are some of them which admit anticonformal involutions. In this case, the previous result can also be obtained as a consequence of the results of [5].

Theorem 2. *Let $\lambda \in \Omega$. In the algebraic model of Theorem 1 the full automorphism group of S_λ is generated by the transformations*

$$r(x,y) = (\omega_q x, \omega_{2q} y) \quad \textit{and} \quad s(x,y) = (\tfrac{1}{x}, \tfrac{y}{x^{q+1}})$$

where $\omega_t = exp(\frac{2\pi i}{t})$.

Proof. The fact that the transformations r and s are indeed automorphisms of S_λ follows from an easy computation. Note that s has order two, r has order $2q$ and

$$sr(x,y) = (\tfrac{1}{\omega_q x}, \tfrac{y}{\omega_{2q} x^{q+1}})$$

has order two; thus, r and s generate a group of order $4q$ isomorphic to $\mathbf{D}_{2q}$.

The proof of the theorem follows after noticing that the stabilizer subgroup of each ramification point of the regular covering map associated to the action of $\langle r, s\rangle$ is conjugate to the group generated by either s, sr^{-2}, r^q or r^{q+2}. In fact:

(a) each power of r^{q+2} has two fixed points with stabilizer subgroup $\langle r\rangle$,
(b) r^q has $2q$ fixed points with stabilizer subgroup $\langle r^q\rangle$,
(c) if t is odd then the involution sr^t does not have fixed points, and

(d) if t is even then the involution sr^t has four fixed points; the G-orbit of each of them has cardinality $2q$, and the stabilizer subgroup of each point in this orbit is conjugate to $\langle s \rangle$.

The proof is done. □

We anticipate the fact that the Jacobian variety JS_λ decomposes, up to isogeny, as a product of an elliptic curve E_λ and (two copies of) the Jacobian of a Riemann surface of genus $\frac{q-1}{2}$.

The next proposition describes algebraically the elliptic curve E_λ.

Proposition 1. *Let $\lambda \in \Omega$, and consider the following subgroup of G*

$$H_4 = \langle r^{-2}, sr^{-1} \rangle \cong \mathbf{D}_q.$$

Then the quotient Riemann surface E_λ given by the action of H_4 on S_λ has genus one, and it is endowed with a two-fold regular covering map over the projective line which ramifies over $\infty, 0, 1$ and λ. In particular, E_λ is isomorphic to the Riemann surface defined by the elliptic curve

$$y^2 = x(x-1)(x-\lambda).$$

Proof. The normality of H_4 as a subgroup of G implies that the quotient group $H := G/H_4 \cong \mathbb{Z}_2$ acts on E_λ. Let

$$\pi_1 : S_\lambda \to E_\lambda \quad \text{and} \quad \pi_2 : E_\lambda \to \mathbb{P}^1$$

denote the branched regular covering maps given by the action of H_4 on S_λ, and by the action of H on E_λ respectively. The following diagram commutes:

$$\begin{array}{ccc} & S_\lambda & \\ {}^{\pi_G}\swarrow & & \searrow^{\pi_1} \\ \mathbb{P}^1 & \xleftarrow{\ \pi_2\ } & E_\lambda \end{array}$$

Following the same notations used in the proof of Theorem 1, we can assert that:

(a) the $2q$ different points $\beta'_1, \dots, \beta'_{2q}$ ($\gamma'_1, \dots, \gamma'_{2q}$ and $\alpha'_1, \dots, \alpha'_{2q}$, respectively) lying over ∞ (over 0 and over 1, respectively) by π_G are not ramification points of π_1 and therefore they are sent to one point in E_λ. Thereby, ∞ (0 and 1, respectively) is a branch value of π_2 marked with two,

(b) the two different points α'_{2q+1} and α'_{2q+2} lying over λ by π_G are ramification points of π_1; the intersection of their stabilizer subgroup with H_4 having order q. Thus, they are sent to one point in E_λ, and λ is a branch value of π_2 marked with two.

Thus, E_λ is endowed with a two-fold regular covering map over the projective line, with four branch values. As the genus of E_λ is one (Riemann–Hurwitz formula), the result follows. □

4. The group algebra decomposition of JS

In this section we consider the Jacobian variety JS_λ and study the group algebra decomposition of it with respect to its full automorphism group G. In order to state the results, we start by studying the representations of G and the generating vectors representing the action of G on S_λ.

It is well-known that the dihedral group

$$G = \langle r, s : r^{2q} = s^2 = (sr)^2 = 1\rangle$$

has, up to equivalence, 4 complex irreducible representations of degree one; namely,

$$V_1 : \begin{cases} r \to 1 \\ s \to 1 \end{cases} V_2 : \begin{cases} r \to 1 \\ s \to -1 \end{cases} V_3 : \begin{cases} r \to -1 \\ s \to 1 \end{cases} V_4 : \begin{cases} r \to -1 \\ s \to -1 \end{cases}$$

and $q-1$ complex irreducible representations of degree two; namely,

$$V_{k+4} : r \mapsto \text{diag}(\omega_{2q}^k, \bar{\omega}_{2q}^k), \;\; s \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$$

for $1 \le k \le q-1$ and $\omega_t = \exp(\frac{2\pi i}{t})$.

Lemma 1.

(1) *The rational irreducible representations of* G*, up to equivalence, are:*
   (a) *four of degree 1; namely* $W_i := V_i$ *for* $1 \le i \le 4$ *and*
   (b) *two of degree* $q-1$*; namely*

$$W_5 = \oplus_{\sigma \in G_5} V_5^{\sigma} \quad \textit{and} \quad W_6 = \oplus_{\sigma \in G_6} V_6^{\sigma}$$

   where G_5 *and* G_6 *denote the Galois group associated to the extensions* $\mathbb{Q} \le \mathbb{Q}(\omega_{2q} + \bar{\omega}_{2q})$ *and* $\mathbb{Q} \le \mathbb{Q}(\omega_q + \bar{\omega}_q)$ *respectively, and* $\omega_t = exp(\frac{2\pi i}{t})$*.*

(2) *Let* $\lambda \in \Omega$*. The group algebra decomposition of* JS_λ *with respect to* G *is*

$$JS_\lambda \sim_G B_1 \times B_2 \times B_3 \times B_4 \times B_5^2 \times B_6^2$$

where B_j *stands for the factor associated to the representation* W_j*.*

Proof. The proof of part (1) follows directly from the way in which the rational irreducible representations of a group are constructed (see Subsection 2.5). The proof of part (2) is a direct consequence of (1) together with the group algebra decomposition theorem (see Subsection 2.6). □

As anticipated in the introduction of this article, to compute the dimension of the factors B_j (which may be zero) we need to choose a generating vector representing the action of G on S_λ; the following lemma provides all those possible choices.

Lemma 2. *Let* σ *be a generating vector of* G *of type* $(2, 2, 2, 2q)$*. Then there exist integers* e_1, e_2 *with* $e_1 - e_2$ *even and not congruent to* 0 *modulo* $2q$*, such that*

$$\sigma = (sr^{e_1}, sr^{e_2}, r^q, r^{e_1 - e_2 + q})$$

up to the action of the symmetric group $\mathbf{S}_3$ *over the first three entries.*

Proof. Let us suppose that

$$\sigma = (g_1, g_2, g_3, g_4 = (g_1 g_2 g_3)^{-1})$$

is a generating vector of G of type $(2, 2, 2, 2q)$. It is not difficult to see that G has exactly three conjugacy classes of elements of order two; namely

$$C_1 = \{sr^n : 0 \leq n < 2q \text{ even}\}, \quad C_2 = \{sr^m : 1 \leq m < 2q \text{ odd}\}$$

and $C_3 = \{r^q\}$. Moreover, there are $\frac{q-1}{2}$ conjugacy classes of elements of G of order $2q$; namely $\{r^t, r^{-t}\}$ for each odd integer $1 \leq t < q$.

Since g_1, g_2 and g_3 must generate G and since their product g_4^{-1} must have order $2q$, it is straightforward to see that:

(a) the elements g_1, g_2 and g_3 cannot belong simultaneously to only one of the conjugacy classes C_1, C_2 or C_3,
(b) the elements g_1, g_2 and g_3 cannot belong to three different conjugacy classes C_1, C_2 and C_3, and
(c) one (and only one) of the elements g_1, g_2 or g_3 must belong to the conjugacy class C_3.

Hence, up to a permutation in $\mathbf{S}_3$, we may assume σ to be of the form

$$(sr^{e_1}, sr^{e_2}, r^q, r^{e_1-e_2+q})$$

where $0 \leq e_1, e_2 < 2q$ are simultaneously odd or simultaneously even. As $e_1 - e_2 + q$ must be coprime with $2q$, the difference $e_1 - e_2$ is not congruent with zero modulo $2q$. The proof is done. □

Remark 2.

(a) We should mention that the proof of the previous lemma could be derived from the proof of Theorem 7 in [6].
(b) Following [6, Remark 8] there is a unique topological class of action of $\mathbf{D}_{2q}$ on Riemann surfaces of genus q with signature $(2, 2, 2, 2q)$; consequently, every generating vector of G of the desired type can be chosen to represent the action of G on S_λ.

We now proceed to analyze how the choice of the generating vector changes the dimension of the factors arising in the group algebra decomposition of JS_λ with respect to G. To accomplish this task, it is convenient to bring in the following equivalence relation:

Definition 1. Two generating vectors σ_1 and σ_2 are termed *essentially equal* with respect to the action of G on S_λ if $\dim_{\tau_1}(B_j) = \dim_{\tau_2}(B_j)$ for all j, where τ_i is the geometric signature associated to σ_i.

Lemma 3. *Each generating vector of G of type $(2, 2, 2, 2q)$ is essentially equal to*

$$\sigma_0 = (s, sr^{-2}, r^q, r^{q+2}) \quad \textit{or to} \quad \sigma_1 = (sr, sr^{-1}, r^q, r^{q+2}).$$

Proof. Let σ be a generating vector of G of type $(2, 2, 2, 2q)$ for the action of G on S_λ. By Lemma 2 we can suppose

$$\sigma = (sr^{e_1}, sr^{e_2}, r^q, r^{e_1-e_2+q})$$

for some integers e_1, e_2 with $e_1 - e_2$ even and not congruent to 0 modulo $2q$, up to the action of $\mathbf{S}_3$ on the first three entries. The action of $\iota \in \mathbf{S}_3$ over the three first entries produces the following change on the fourth one:

$$r^{e_1-e_2+q} \mapsto \iota(r^{e_1-e_2+q}) = r^{\pm(e_1-e_2+q)}$$

sending $r^{e_1-e_2+q}$ into either itself or its inverse. Hence, in spite of the fact that the corresponding geometric signature changes under permutations in $\mathbf{S}_3$, the dimension of the each factor B_j remains the same (ι permutes the summands in the sum (2.7)); thus the generating vectors σ and $\iota(\sigma)$ are essentially equal.

We remark the obvious observation that the geometric signature associated to a given generating vector is kept invariant under inner automorphisms of the group. Now, after conjugating every element in σ by

$$\begin{cases} r^{-e_1/2} & \text{if } e_1 \text{ and } e_2 \text{ are even;} \\ r^{-(e_1+1)/2} & \text{if } e_1 \text{ and } e_2 \text{ are odd,} \end{cases}$$

we obtain normalized generating vectors

$$\sigma_{0,n} := (s, sr^{-n}, r^q, r^{q+n}) \quad \text{and} \quad \sigma_{1,n} := (sr, sr^{1-n}, r^q, r^{q+n})$$

for e_1, e_2 even, and for e_1, e_2 odd respectively, where $n = e_1 - e_2$.

Note that if n and m are distinct even numbers, then $\sigma_{0,n}$ and $\sigma_{0,m}$ are essentially equal, and $\sigma_{1,n}$ and $\sigma_{1,m}$ are essentially equal.

Hence, the result follows after verifying that $\sigma_0 = \sigma_{0,2}$ and $\sigma_1 = \sigma_{1,2}$ are not essentially equal; this follows from $\dim_{\tau_0}(B_3) = 0$ and $\dim_{\tau_1}(B_3) = 1$. □

Proposition 2. *Let $\lambda \in \Omega$, and consider the group algebra decomposition of JS_λ with respect to G*

$$JS_\lambda \sim_G B_1 \times B_2 \times B_3 \times B_4 \times B_5^2 \times B_6^2.$$

If τ_0 denotes the geometric signature associated to $\sigma_0 = (s, sr^{-2}, r^q, r^{q+2})$ then

$$dim_{\tau_0}(B_j) = \begin{cases} 0 & \text{if } j = 0,1,2,3,6 \\ 1 & \text{if } j = 4 \\ \frac{q-1}{2} & \text{if } j = 5 \end{cases}$$

If τ_1 denotes the geometric signature associated to $\sigma_1 = (sr, sr^{-1}, r^q, r^{q+2})$, then

$$dim_{\tau_0}(B_j) = \begin{cases} 0 & \text{if } j = 0,1,2,4,6 \\ 1 & \text{if } j = 3 \\ \frac{q-1}{2} & \text{if } j = 5 \end{cases}$$

In particular, JS_λ contains an elliptic curve.

Proof. The genus of the quotient S/G is zero; thus, $\dim_{\tau_0}(B_1) = \dim_{\tau_1}(B_1) = 0$. The table below summarized the dimension of the vector subspaces of each V_j fixed under the cyclic subgroup $\langle g \rangle$, for each g arising in the signatures σ_0 and σ_1.

	$\langle s \rangle$	$\langle sr \rangle$	$\langle sr^{-2} \rangle$	$\langle sr^{-1} \rangle$	$\langle r^q \rangle$	$\langle r^{q+2} \rangle$
V_2	0	0	0	0	1	1
V_3	1	0	1	0	0	0
V_4	0	1	0	1	0	0
V_5	1	1	1	1	0	0
V_6	1	1	1	1	2	0

Now, the result follows directly as an application of (2.7). □

Theorem 3. *Let $\lambda \in \Omega$. The group algebra decomposition of JS_λ with respect to G does not depend on the choice of the generating vector.*

Proof. By Lemmata 2 and 3, we only need to compare the decompositions associated to σ_0 and σ_1. By Proposition 2, these decompositions are

$$JS_\lambda \sim_{G,\sigma_0} B_4 \times B_5^2 \quad \text{and} \quad JS_\lambda \sim_{G,\sigma_1} B_3 \times B_5^2$$

respectively, showing that B_3 and B_4 are isogenous. We claim that, in addition, B_4 and B_5 are equal. Indeed, note that the generating vectors σ_0 and σ_1 are identified by the action of the outer automorphism Φ of G defined by $r \mapsto r, s \mapsto sr$.

Accordingly, at the level of rational irreducible representations, W_3 is sent by Φ to W_4. As a matter of fact, this shows that the roles played by B_3 and B_4 are interchanged according to the choice of the generating vector employed to compute the dimensions.

The proof is done. □

Remark 3. The independence of the group algebra decomposition on the choice of the generating vector when there is a unique topological class of action is not new and was firstly noticed by Rojas in [44, Example 4.3] when she considered the Weyl group $\mathbb{Z}_2^3 \rtimes \mathbf{S}_3$ acting on a Riemann surface of genus three with signature $(2, 4, 6)$.

Very recently, the same phenomenon has been noticed by Izquierdo, Jiménez and Rojas itself in [29] when they studied a two-dimensional family of Riemann surfaces of genus $2n - 1$ with action of $\mathbf{D}_{2n}$ with signature $(2, 2, 2, 2, n)$.

It is worth recalling that in the two aforementioned cases as well as in the case of the family $\mathcal{F}_q$, the existence of outer automorphisms of the group is the key ingredient. Based on the evidence of explicit examples, it seems reasonable to ask if this is the general situation; however, according to the knowledge of the author, it has not been proved a general result on this respect.

From now on, we assume the action of G on S_λ to be determined by the generating vector σ_0 and, in consequence, the group algebra decomposition of JS_λ with respect to G to be of the form

$$JS_\lambda \sim_G B_4 \times B_5^2.$$

The following result shows that the factors B_4 and B_5 have a geometric meaning.

Theorem 4. *Let $\lambda \in \Omega$. Consider the subgroups $H_4 = \langle r^{-2}, sr^{-1} \rangle$ and $H_5 = \langle s \rangle$ of G, and the quotient Riemann surfaces E_λ and C_λ given by the action of H_4 and of H_5 on S_λ, respectively. Then*

$$B_4 \sim JE_\lambda \quad \text{and} \quad B_5 \sim JC_\lambda.$$

In particular, JS_λ decomposes into a product of Jacobians as follows:

$$JS_\lambda \sim_G JE_\lambda \times JC_\lambda^2.$$

Proof. The dimension of the complex vector subspaces of V_4 and V_5 fixed under the subgroups H_4 and H_5 are

$$d_{V_4}^{H_4} = d_{V_5}^{H_5} = 1 \quad \text{and} \quad d_{V_4}^{H_5} = d_{V_5}^{H_4} = 0.$$

Thus, the result follows after applying the criterion to identify factors in the group algebra decomposition of JS_λ as Jacobians of quotients of S_λ (as explained in Subsection 2.6; see equations (2.5) and (2.6)) together with Proposition 2. □

Remark 4. Note that C_λ is an irregular $2q$-gonal Riemann surface of genus $\frac{q-1}{2}$. An explicit algebraic description of E_λ has been obtained in Proposition 1.

5. Fields of definition

Let Gal($\mathbb{C}$) denote the group of field automorphisms of $\mathbb{C}$. Let $X \subset \mathbb{P}^n$ be a (smooth algebraic) variety and $\sigma \in \text{Gal}(\mathbb{C})$. We shall denote by X^σ the variety defined by the polynomials obtained after applying σ to the coefficients of the polynomials which define X.

Let k be a subfield of $\mathbb{C}$ and let Gal($\mathbb{C}/k$) be the subgroup of Gal($\mathbb{C}$) consisting of those automorphisms which fix the elements in k. We shall say that X *is defined over* k if $X = X^\sigma$ for all $\sigma \in \text{Gal}(\mathbb{C}/k)$. We shall say that X *can be defined over* k (or that k *is a field of definition for* X) if there exists an isomorphism $X \to Y$ into a variety $Y \subset \mathbb{P}^m$ which is defined over k.

By considering the explicit algebraic description of S_λ provided in Theorem 1, in this section we derive results concerning the field of definitions of S_λ according to the value of λ.

5.1. Real Riemann surfaces

An algebraic variety is called *real* if it can be defined over the field of the real numbers; equivalently, if it admits an anticonformal involution (i.e. an anticonformal automorphism of order two).

Following [6, Section 6], when we consider the family $\mathcal{F}_q$ as a complex subvariety of the moduli space $\mathscr{M}_q$ of compact Riemann surfaces of genus q, it is isomorphic to the projective line minus three points. Furthermore, $\mathcal{F}_q \subset \mathscr{M}_q$ admits an anticonformal involution whose fixed point set consists of points representing real Riemann surfaces.

The following result shows that among the Riemann surfaces S_λ in $\mathcal{F}_q$, the real ones can be easily recognized according to the value of λ. More precisely:

Theorem 5. *Let $\lambda \in \Omega$. Then the following statements are equivalent:*

(a) *S_λ is a real Riemann surface.*
(b) *JS_λ is a real algebraic variety.*
(c) $\lambda \in \{\bar{\lambda}, 1-\bar{\lambda}, 1/\bar{\lambda}, \bar{\lambda}/(1-\bar{\lambda})\}$

Proof. The equivalence between the first two statements is well-known; indeed, following [37, Theorem 1.1], a Riemann surface and its Jacobian variety can be defined over the same fields.

We now proceed to prove the equivalence between the statements (a) and (c).

Let us assume that S_λ is a real Riemann surface or, equivalently, that S_λ admits an anticonformal involution, denoted by $f_\lambda : S_\lambda \to S_\lambda$. It is clear that $f_\lambda G f_\lambda^{-1} = G$ and therefore f_λ gives rise to an anticonformal involution $g_\lambda : \mathcal{O}_\lambda \to \mathcal{O}_\lambda$, where $\mathcal{O}_\lambda$ denotes the Riemann orbifold given by the action of G on S_λ.

We recall that $\mathcal{O}_\lambda$ has genus zero and four marked points: $0, 1$ and ∞ marked with 2, and λ marked with $2q$. It follows that g_λ is an extended Möbius transformation, i.e. $g_\lambda(z) = (a\bar{z}+b)/(c\bar{z}+d)$ with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$, satisfying

$$g_\lambda(\lambda) = \lambda \quad \text{and} \quad g_\lambda(\{\infty, 0, 1\}) = \{\infty, 0, 1\}.$$

We only have four possibilities:

(1) g_λ fixes ∞ and permutes 0 and 1. In this case $g_\lambda(z) = 1 - \bar{z}$ showing that $\lambda = 1 - \bar{\lambda}$.
(2) g_λ fixes 0 and permutes 1 and ∞. In this case $g_\lambda(z) = \frac{\bar{z}}{1-\bar{z}}$ showing that $\lambda = \bar{\lambda}/(1 - \bar{\lambda})$.
(3) g_λ fixes 1 and permutes ∞ and 0. In this case $g_\lambda(z) = \frac{1}{\bar{z}}$ showing that $\lambda = 1/\bar{\lambda}$.
(4) g_λ fixes $\infty, 0$ and 1. In this case $g_\lambda(z) = \bar{z}$ showing that $\lambda = \bar{\lambda}$.

Hence, λ is as in statement (c).

Conversely, if λ is as in statement (c), to construct explicitly an anticonformal involution is an easy task, and the proof is done. □

Remark 5. Following [6, Theorem 14], the real Riemann surfaces in the family $\mathcal{F}_q$ form three (one-real-dimensional) arcs. In addition, in order to compactify the union of these arcs in the Deligne–Mumford compactification of $\mathscr{M}_g$, it was proved that it is enough to add to $\mathcal{F}_q$ three points: these points representing two nodal Riemann surfaces, and the *Wiman surface* of type II (this is the unique compact Riemann surface of genus q admitting an automorphism of order $4q$; see [52]).

The aforementioned results were obtained by using Teichmüller theory and Fuchsian groups, among other techniques. Here, by considering the algebraic description of the Riemann surfaces in $\mathcal{F}_q$ given in Theorem 1, we are able to recover partly these results in a very explicit way as follows.

The Riemann surfaces S_{λ_1} and S_{λ_2} are isomorphic if and only if $\lambda_2 = T(\lambda_1)$ for some

$$T \in \mathbb{G} = \langle z \mapsto \tfrac{1}{z}, z \mapsto \tfrac{1}{1-z} \rangle \cong \mathbf{S}_3. \tag{5.1}$$

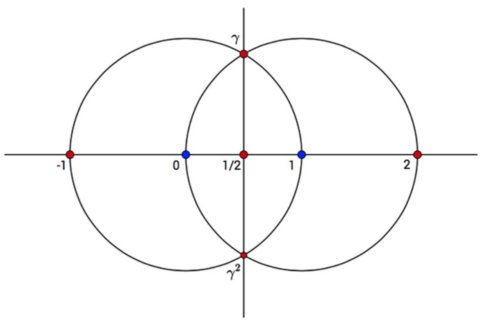
Observe that for the exceptional values $-1, \frac{1}{2}, 2, \gamma$ and γ^2 where $\gamma^3 = -1$, the Riemann surface S_λ has more than $4q$ automorphisms.

Thus, the family $\mathcal{F}_q$ is isomorphic to the quotient of the parameter space

$$\Omega = \mathbb{C} - \{0, \pm 1, \tfrac{1}{2}, 2, \gamma, \gamma^2\}$$

up to the action of $\mathbb{G}$. Namely: $\Omega \to \Omega/\mathbb{G} \cong \mathcal{F}_q \cong \mathbb{C} - \{0, 1\}$.

According to Theorem 5, the complex numbers $\lambda \in \Omega$ representing Riemann surfaces S_λ which are real can be represented in the diagram below; the colored red points represent Riemann surfaces with more than $4q$ automorphisms (and therefore they do not belong to $\mathcal{F}_q$).



Note that a fundamental region for the action of $\mathbb{G}$ on Ω is given by

$$\{z \in \mathbb{C} : |z| < 1, \mathrm{Re}(z) < \tfrac{1}{2}\}$$

and, consequently, the subsets of $\mathcal{F}_q$ given by

$$\Pi(\{e^{i\theta} : \pi < \theta < \tfrac{\pi}{2}\}),\ \Pi(\{z : |z-1| = 1, |z| < 1\}) \text{ and } \Pi(]-1,0[)$$

are the three arcs in $\mathcal{F}_q$ (denoted in [6] by a_2, a_1 and b respectively) corresponding to real Riemann surfaces in $\mathcal{F}_q$.

Note that the limit point of $\mathcal{F}_q$ which connects the arcs a_2 and b correspond to S_{-1} and therefore, by Theorem 1, can be algebraically described by

$$y^2 = x(x^{2q} + 1).$$

The map $(x, y) \mapsto (-\omega_{4q}x, \omega_{8q}y)$ where $\omega_t = \exp(\frac{2\pi i}{t})$, induces an isomorphism between S_{-1} and the curve

$$y^2 = x(x^{2q} - 1);$$

this is the Wiman surface of type II.

Remark 6. After proving that an algebraic variety is real, to find explicit defining equations with real coefficients is, in general, a difficult problem. If λ is real then a model for S_λ in terms of equations with real coefficients is provided by Theorem 1. In the remaining cases, the construction of real equations can be done by applying the results of [25].

5.2. Arithmetic Riemann surfaces

An algebraic variety is called *arithmetic* if it can be defined over a number field or, equivalently, over the algebraic closure $\overline{\mathbb{Q}}$ of the field of the rational numbers.

A well-known result due to Belyi ensures that a Riemann surface is arithmetic if and only if it admits a non-constant meromorphic function with three critical values; see [2]. For arithmetic complex surfaces an analogous result to Belyi's theorem was proved by González-Diez in [16] by considering the so-called Lefschetz maps. For the case of arithmetic families of Riemann surfaces we refer to the articles [18] and [19].

We mention that arithmetic Riemann surfaces (also known as *Belyi curves*) have attracted much attention ever since Grothendieck noticed, in his famous Esquisse d'un Programme, interesting relations between them and bipartite graphs embedded in a topological surface; see [20].

As in the case of real Riemann surfaces, arithmetic Riemann surfaces among the Riemann surfaces in the family $\mathcal{F}_q$ can be easily identified.

Theorem 6. *Let $\lambda \in \Omega$. Then the following statements are equivalent:*

(a) *S_λ is an arithmetic Riemann surface.*
(b) *JS_λ is an arithmetic algebraic variety.*
(c) *λ is an algebraic complex number.*

Proof. As in the proof of Theorem 5, the equivalence between the first two statements follows from [37, Theorem 1.1].

We denote by $\mathcal{O}_\lambda$ the Riemann orbifold given by the action of G_λ on S_λ, and by

$$\pi_{G_\lambda} : S_\lambda \to \mathcal{O}_\lambda$$

the associated covering map.

Let us assume that S_λ is arithmetic. Then there exists an algebraic model S'_λ of S_λ defined by equations whose coefficients belong to the field of the algebraic numbers. Let us denote by G'_λ the automorphism group of S'_λ, by $\mathcal{O}'_\lambda$ the Riemann orbifold given by the action of G'_λ on S'_λ, and by $\pi_{G'_\lambda}$ the associated covering map.

As a consequence of [17, Proposition 3.3], both each element of G'_λ and the projection $\pi_{G'_\lambda}$ are algebraic (i.e. defined over $\bar{\mathbb{Q}}$). In particular, the branch values of $\pi_{G'_\lambda}$ are also algebraic. Let μ_0, μ_1, μ_∞ and μ_λ denote these values, where μ_0, μ_1, μ_∞ are marked with 2 and μ_λ is marked with $2q$.

Now, the existence of an isomorphism $f_\lambda : S_\lambda \to S'_\lambda$ guarantees the existence of an isomorphism $g_\lambda : \mathcal{O}_\lambda \to \mathcal{O}'_\lambda$ such that $\pi_{G'_\lambda} \circ f_\lambda = g_\lambda \circ \pi_{G_\lambda}$. It follows that g_λ is a Möbius transformation satisfying that

$$g_\lambda(\mu_\lambda) = \lambda \quad \text{and} \quad g_\lambda(\{\mu_\infty, \mu_0, \mu_1\}) = \{\infty, 0, 1\}.$$

Thus,

$$g_\lambda(z) = T\left(\tfrac{(\mu_1-\mu_\infty)(z-\mu_0)}{(t_1-t_0)(z-\mu_\infty)}\right)$$

for some $T \in \mathbb{G}$ as in (5.1), and therefore

$$\lambda = T\left(\tfrac{(\mu_1-\mu_\infty)(\mu_\lambda-\mu_0)}{(\mu_1-\mu_0)(\mu_\lambda-\mu_\infty)}\right).$$

Finally, as each $T \in \mathbb{G}$ is defined over $\mathbb{Q}$ and the points $\mu_0, \mu_1, \mu_\infty, \mu_\lambda$ are algebraic, we are in position to conclude that the complex number λ must be algebraic.

The converse follows directly from Theorem 1, and the proof is done. □

Corollary 1. *Let $\lambda \in \Omega$. Then JS_λ is an arithmetic algebraic variety admitting a group algebra decomposition in which each factor is arithmetic as well.*

Proof. Following [17, Theorem 4.4], if S is an arithmetic Riemann surface then any Riemann surface S' for which there is a covering map $S \to S'$ is arithmetic as well. Thus, the result follows directly from Theorems 4 and 6. □

Remark 7.

(a) It is worth observing that Theorem 6 and Corollary 1 can be easily generalized from $\bar{\mathbb{Q}}$ to any algebraically closed subfield k of the field of the complex numbers.
(b) In addition, Corollary 1 can also be generalized from each S_λ in $\mathcal{F}_q$ to any Riemann surface S defined over k whose Jacobian variety admit a group algebra decomposition in which every factor is isogenous to the Jacobian of a quotient of S.

5.3. Riemann surfaces defined over the field of moduli

The *field of moduli* $\mathcal{M}(S)$ of a compact Riemann surface S is by definition the fixed field of the group

$$\mathbb{I}(S) = \{\sigma \in \mathrm{Gal}(\mathbb{C}) : S^\sigma \cong S\}.$$

Proposition 3. *Let $\lambda \in \Omega$. Then*

$$\mathbb{Q}(j(\lambda)) \leq \mathcal{M}(S) \leq \mathbb{Q}(\lambda)$$

where j denotes the invariant function for elliptic curves, in the Legendre form.

Proof. We recall that, as a consequence of Theorem 1 and Proposition 1,

$$(S_\lambda)^\sigma = S_{\sigma(\lambda)} \quad \text{and} \quad (E_\lambda)^\sigma = E_{\sigma(\lambda)}$$

for all $\sigma \in \mathrm{Gal}(\mathbb{C})$, where $E_\lambda = S_\lambda / \langle r^{-2}, sr^{-1}\rangle$.

Now, if $\sigma \in \mathbb{I}(S)$ then there is an isomorphism $S_\lambda \to S_{\sigma(\lambda)}$ which induces an isomorphism $E_\lambda \to E_{\sigma(\lambda)}$. In particular,

$$j(\lambda) = j(\sigma(\lambda)) = \sigma(j(\lambda))$$

showing that $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q}(j(\lambda)))$; it follows that $\mathbb{Q}(j(\lambda)) \leq \mathcal{M}(S)$.

The other inclusion follows from Theorem 1, and from the fact that the field of moduli is contained in every field of definition. The proof is done. □

Weil in [51] provided necessary conditions for S to admit its field of moduli as a field of definition; these conditions hold trivially if S does not have non-trivial automorphisms. On the other extreme, following [53], if $S/\mathrm{Aut}(S)$ is an orbifold with signature of type (a, b, c) then S can be defined over its field of moduli.

By results of Dèbes–Emsalem [13] (see also Hammer–Herrlich [21]) there is a field of definition of S which is an extension of finite degree of its field of moduli.

In general, the determination of whether the field of moduli is a field of definition is a difficult task; see, for example [14], [23], [26], [41] and [50]. By contrast, in the hyperelliptic case it is possible to decide, in a very simple way, if the field of moduli is a field of definition. In fact, following [28], if the reduced automorphism group of a hyperelliptic Riemann surface is not cyclic, then it can be defined over its field of moduli. It follows immediately the following:

Proposition 4. *Let $\lambda \in \Omega$. The field of moduli of S_λ is a field of definition for S_λ and for JS_λ.*

6. A three-dimensional family of ppavs with $\mathbf{D}_{10}$-action

Let S be a compact Riemann surface of genus $g \geq 2$, and let

$$JS = (\mathscr{H}^{1,0}(S, \mathbb{C}))^* / H_1(S, \mathbb{Z})$$

be its Jacobian variety. We recall that, after fixing a symplectic basis of $H_1(S, \mathbb{Z})$, both a period matrix $(I_g\, Z_S)$ with $Z_S \in \mathscr{H}_g$ for JS, and a rational representation of $L_S := \mathrm{End}_{\mathbb{Q}}(JS)$ are determined, up to equivalence.

If S is hyperelliptic, then the symplectic representation

$$\rho_r : G \to \mathrm{Sp}(2g, \mathbb{Z})$$

of the automorphism group G of S induces an isomorphism

$$G \cong \mathcal{G} := \{R \in \mathrm{Sp}(2g, \mathbb{Z}) : R \cdot Z_S = Z_S\}.$$

We can now consider the complex submanifold of $\mathscr{H}_g$

$$\mathscr{H}_g(G) = \{Z \in \mathscr{H}_g : R \cdot Z = Z \text{ for all } R \in \mathcal{G}\}$$

consisting of those period matrices Z representing ppavs of dimension g admitting the given action of G. Clearly, $Z_S \in \mathscr{H}_g(G)$.

In the case of the action of $\mathbf{D}_{10}$ on the Riemann surfaces in family $\mathcal{F}_5$, we can be much more explicit.

Theorem 7. *Consider the action of* $\mathbf{D}_{10}$ *with generating vector* σ_0.

There exists a three-dimensional family $\mathcal{A}_5(\mathbf{D}_{10})$ *of principally polarized abelian varieties of dimension five admitting the given group action; it is given by the period matrices in* $\mathscr{H}_5$ *of the following form:*

$$\begin{pmatrix} 2(u+v+u) & -w-u & -2v & -v-w-u & -v+u \\ -w-u & -v-\frac{1}{2}w+\frac{5}{4}u & v-\frac{1}{2}u & w+\frac{1}{2}u & v-u \\ -2v & v-\frac{1}{2}u & u & v & w \\ -v-w-u & w+\frac{1}{2}u & v & u & -w \\ -v+u & v-u & w & -w & 2(u-v-w) \end{pmatrix} \tag{6.1}$$

for complex numbers u, v *and* w.

Furthermore, $\mathcal{A}_5(\mathbf{D}_{10})$ *contains the one-dimensional family* $\mathcal{F}_5$.

Proof. The proof is based on the results and routines in [1] (implemented in the open source computer algebra system SAGE).

By constructing a family of very special hyperbolic polygons that uniformize Riemann surfaces with a given group action, it was implemented, among others, routines to determine a symplectic representation of the group, and after that, those matrices which are invariant.

We consider the generating vector $\sigma_0 = (s, sr^{-2}, r^5, r^7)$ of $G = \mathbf{D}_{10}$. By applying the routine *P.symplectic.generators*, we obtain that, if ρ denotes the symplectic representation of G, then

$$\rho(r) = \text{diag}(R, (R^t)^{-1}) \quad \text{and} \quad \rho(s) = \text{diag}(S, S^t),$$

where

$$R = \begin{pmatrix} -1 & 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & -1 \\ 0 & -2 & 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} -1 & 0 & 1 & -1 & 1 \\ 0 & -1 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The problem of finding those period matrices in $\mathscr{H}_5$ which are invariant under the given action involves solving a system of nonlinear equations in fifteen variables. If we apply the routine *P.moebius.invariant*, the desired form is obtained. □

The automorphism group G of S can be canonically seen as a subgroup of L_S. Thus, the variety $\mathscr{H}_g(G)$ contains the complex submanifold $\mathbb{H}(L_S)$ whose points are matrices representing ppavs containing L_S in their endomorphism algebras; see [54, Section 3] and also [45, Sections 2 and 3] for a more general context. This is called the *Shimura family* of S and corresponds to a *special subvariety* of $\mathcal{A}_g$ (see [38, Section 3] for a precise definition).

Proposition 5. *Let* $\lambda \in \Omega$. *The dimension of the Shimura family of each Riemann surface* S_λ *in* $\mathcal{F}_q$ *is* $\frac{q+1}{2}$.

Proof. Following the results proved in [15] and Serre's formula [47, Proposition 3], it can be seen that the dimension N of the Shimura family of S_λ is given by

$$\tfrac{1}{8q}\Sigma_{g \in G}[\chi(g)^2 + \chi(g^2)],$$

where χ stands for the character of the analytic representation ρ_a of G. Clearly, this dimension does not depend on λ; in fact, it only depends on the local monodromy of the action of G on S_λ.

Now, by using the classically known Chevalley–Weil formula [10], we obtain that

$$\rho_a \cong W_4 \oplus W_5.$$

The character of ρ_a is summarized in the following table:

Rep. of conj. class	id	s	sr	r^q	r^t
Length	1	q	q	1	2
Character	q	-1	1	$-q$	0

where $1 \leq t \leq q-1$. It follows that

$$N = \tfrac{1}{8q}[(q^2+q)+(1+q)q+(1+q)q+(q^2+q)] = \tfrac{q+1}{2}. \quad \square$$

Given a Riemann surface S, to provide an explicit description of the elements of $\mathbb{H}(L_S)$ seems to be a difficult task. However, as a simple consequence of Theorem 7, we obtain the following direct corollary:

Corollary 2. *Each element of the Shimura family associated to every member of the family $\mathcal{F}_5$ admits a period matrix of the form (6.1) for some $u, v, w \in \mathbb{C}$.*

References

[1] A. Behn, R.E. Rodríguez, A.M. Rojas, Adapted hyperbolic polygons and symplectic representations for group actions on Riemann surfaces, J. Pure Appl. Algebra 217 (3) (2013) 409–426.
[2] G. Belyi, On Galois extensions of a maximal cyclotomic field, Math. USSR, Izv. 14 (1980) 247–265.
[3] Ch. Birkenhake, H. Lange, Complex Abelian Varieties, 2nd edition, Grundl. Math. Wiss., vol. 302, Springer, 2004.
[4] S.A. Broughton, Finite groups actions on surfaces of low genus, J. Pure Appl. Algebra 69 (3) (1991) 233–270.
[5] E. Bujalance, F.J. Cirré, J.M. Gamboa, G. Gromadzki, Symmetry types of hyperelliptic Riemann surfaces, Mém. Soc. Math. Fr. (N.S.) 86 (2001), vi+122 pp.
[6] E. Bujalance, A.F. Costa, M. Izquierdo, On Riemann surfaces of genus g with 4g automorphisms, Topol. Appl. 218 (2017) 1–18.
[7] A. Carocca, S. Recillas, R.E. Rodríguez, Dihedral groups acting on Jacobians, Contemp. Math. 311 (2011) 41–77.
[8] A. Carocca, R.E. Rodríguez, Jacobians with group actions and rational idempotents, J. Algebra 306 (2) (2006) 322–343.
[9] M. Carvacho, R.A. Hidalgo, S. Quispe, Jacobian variety of generalized Fermat curves, Q. J. Math. 67 (2) (2016) 261–284.
[10] C. Chevalley, A. Weil, Über das Verhalten der Integrale erster Gattung bei Automorphismen des Funktionenkörpers, Hamb. Abh. 10 (1934) 358–361.
[11] A.F. Costa, M. Izquierdo, One-dimensional families of Riemann surfaces of genus g with $4g+4$ automorphims, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. (2018), https://doi.org/10.1007/s13398-017-0429-0.
[12] O. Debarre, Tores et variétés abéliennes complexes, Cours Spécialisés, vol. 6, Société Mathématique de France, Paris, 1999, EDP Sciences, Les Ulis.
[13] P. Dèbes, M. Emsalem, On fields of moduli of curves, J. Algebra 211 (1999) 42–56.
[14] C.J. Earle, On the moduli of closed Riemann surfaces with symmetries, in: L.V. Ahlfors, et al. (Eds.), Advances in the Theory of Riemann Surfaces, Princeton Univ. Press, Princeton, 1971, pp. 119–130.
[15] P. Frediani, A. Ghigi, M. Penegini, Shimura varieties in the Torelli locus via Galois coverings, Int. Math. Res. Not. 20 (2015) 10595–10623.
[16] G. González-Diez, Belyi's theorem for complex surfaces, Am. J. Math. 130 (1) (2008) 59–74.
[17] G. González-Diez, Variations on Belyi's theorem, Q. J. Math. 57 (2006) 339–354.
[18] G. González-Diez, S. Reyes-Carocca, Families of Riemann surfaces, uniformization and arithmeticity, Trans. Am. Math. Soc. 370 (3) (2018) 1529–1549.
[19] G. González-Diez, S. Reyes-Carocca, The arithmeticity of a Kodaira fibration is determined by its universal cover, Comment. Math. Helv. 90 (2) (2015) 429–434.
[20] A. Grothendieck, Esquisse d'un Programme, in: L. Schneps, P. Lochak (Eds.), Geometric Galois Actions 1, Around Grothendieck's Esquisse d'un Programme, in: London Math. Soc. Lecture Note Ser., vol. 242, 1997, pp. 5–48.
[21] H. Hammer, F. Herrlich, A remark on the moduli field of a curve, Arch. Math. 81 (2003) 5–10.
[22] J. Harvey, On branch loci in Teichmüller space, Trans. Am. Math. Soc. 153 (1971) 387–399.
[23] R.A. Hidalgo, Non-hyperelliptic Riemann surfaces with real field of moduli but not definable over the reals, Arch. Math. 93 (2009) 219–222.
[24] R.A. Hidalgo, L. Jiménez, S. Quispe, S. Reyes-Carocca, Quasiplatonic curves with symmetry group $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_m$ are definable over $\mathbb{Q}$, Bull. Lond. Math. Soc. 49 (2017) 165–183.
[25] R.A. Hidalgo, S. Reyes-Carocca, A constructive proof of Weil's Galois descent theorem, arXiv:1203.6294.
[26] R.A. Hidalgo, S. Reyes-Carocca, Fields of moduli of classical Humbert curves, Q. J. Math. 63 (4) (2012) 919–930.
[27] J. Hubbard, Teichmüller Theory and Applications to Geometry, Topology, and Dynamics, vol. 1, Matrix Editions, Ithaca, New York, 2006.
[28] B. Huggins, Fields of moduli of hyperelliptic curves, Math. Res. Lett. 12 (2) (2007) 249–262.
[29] M. Izquierdo, L. Jiménez, A. Rojas, Decomposition of Jacobian varieties of curves with dihedral actions via equisymmetric stratification, arXiv:1609.01562.

[30] L. Jiménez, On the group algebra decomposition of a Jacobian variety, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 110 (1) (2016) 185–199.
[31] R. Kani, M. Rosen, Idempotent relations and factors of Jacobians, Math. Ann. 284 (1989) 307–327.
[32] R.S. Kulkarni, A note on Wiman and Accola–Maclachlan surfaces, Ann. Acad. Sci. Fenn., Ser. A 1 Math. 16 (1) (1991) 83–94.
[33] R.S. Kulkarni, Riemann surfaces admitting large automorphism groups, in: Extremal Riemann Surfaces, San Francisco, CA, 1995, in: Contemp. Math., vol. 201, Amer. Math. Soc., Providence, RI, 1997, pp. 63–79.
[34] H. Lange, S. Recillas, Abelian varieties with group actions, J. Reine Angew. Math. 575 (2004) 135–155.
[35] A.M. McBeath, The classification of non-Euclidean crystallographic groups, Can. J. Math. 19 (1966) 1192–1205.
[36] A.M. McBeath, Generators of the linear fractional groups, in: Number Theory, Proc. Sympos. Pure Math., vol. XII, Houston, Tex., 1967, 1969, pp. 14–32.
[37] J.S. Milne, Jacobian varieties, in: G. Cornell, J.H. Silverman (Eds.), Arithmetic Geometry, Springer-Verlag, 1986, pp. 167–212.
[38] B. Moonen, F. Oort, The Torelli locus and special subvarieties, in: Handbook of Moduli: Volume II, International Press, Boston, MA, 2013, pp. 549–594.
[39] K. Nakagawa, On the orders of automorphisms of a closed Riemann surface, Pac. J. Math. 115 (2) (1984) 435–443.
[40] J. Paulhus, A.M. Rojas, Completely decomposable Jacobian varieties in new genera, Exp. Math. 26 (4) (2017) 430–445.
[41] S. Reyes-Carocca, Field of moduli of generalized Fermat curves, Q. J. Math. 63 (2) (2012) 467–475.
[42] S. Reyes-Carocca, R.E. Rodríguez, A generalisation of Kani–Rosen decomposition theorem for Jacobian varieties, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (2018), https://doi.org/10.2422/2036-2145.201706-003, in press, arXiv:1702.00484.
[43] S. Reyes-Carocca, R.E. Rodríguez, On Jacobians with group action and coverings, arXiv:1711.07552.
[44] A.M. Rojas, Group actions on Jacobian varieties, Rev. Mat. Iberoam. 23 (2007) 397–420.
[45] B. Runge, On algebraic families of polarized abelian varieties, Abh. Math. Semin. Univ. Hamb. 69 (1999) 237–258.
[46] A. Sánchez-Argáez, Actions of the Group A_5 in Jacobian Varieties, Aportaciones Mat. Comun., vol. 25, Soc. Mat. Mexicana, México, 1999, pp. 99–108.
[47] J.P. Serre, Linear Representations of Finite Groups, Graduate Texts in Maths, vol. 42.
[48] D. Singerman, Finitely maximal Fuchsian groups, J. Lond. Math. Soc. (2) 6 (1972) 29–38.
[49] D. Singerman, Subgroups of Fuchsian groups and finite permutation groups, Bull. Lond. Math. Soc. 2 (1970) 319–323.
[50] G. Shimura, On the field of rationality for an abelian variety, Nagoya Math. J. 45 (1972) 167–178.
[51] A. Weil, The field of definition of a variety, Am. J. Math. 78 (1956) 509–524.
[52] A. Wiman, Über die hyperelliptischen Curven und diejenigen von Geschlechte p – Jwelche eindeutige Tiansformationen in sich zulassen, Bih. K. Sven. Vet.-Akad. Handl., Stockh. 21 (1895–1896) 1–28.
[53] J. Wolfart, *ABC* for polynomials, dessins d'enfants and uniformization – a survey, in: Elementare und analytische Zahlentheorie, in: Schr. Wiss. Ges. Johann Wolfgang Goethe Univ. Frankfurt am Main, vol. 20, Franz Steiner Verlag Stuttgart, Stuttgart, 2006, pp. 313–345.
[54] J. Wolfart, Regular dessins, endomorphisms of Jacobians and transcendence, in: G. Wüstholz (Ed.), A Panorama of Number Theory or the View from Baker's Garden, Cambridge University Press, Cambridge, 2002, pp. 107–120.