Journal of Mathematical Analysis and Applications **267**, 760–774 (2002)
doi:10.1006/jmaa.2001.7831, available online at http://www.idealibrary.com on IDEAL®

Stability Results for the First Eigenvalue of the Laplacian on Domains in Space Forms[1]

Andrés I. Ávila[2]

Department of Mathematics, University of Missouri-Columbia,
Columbia, Missouri 65211
E-mail: andres@banach.math.missouri.edu

Submitted by Karen A. Ames

Received December 13, 1999

We studied the two known works on stability for isoperimetric inequalities of the first eigenvalue of the Laplacian. The earliest work is due to A. Melas who proved the stability of the Faber–Krahn inequality: for a convex domain Ω contained in $\mathbb{R}^n$ with λ close to $\bar{\lambda}$, the first eigenvalue of the ball B of the same volume, the domain must be close to the ball B with respect to the Hausdorff distance. Later, Y. Xu studied the stability of the Szegö–Weinberger inequality for convex domains in $\mathbb{R}^n$ and $\mathbb{H}^n$ where $\mathbb{H}^n$ denotes hyperbolic space. Our work consists of extending A. Melas' result to the spaces of constant curvature $\mathbb{S}^2$ and $\mathbb{H}^2$ and Y. Xu's result to domains contained in the polar cap $B_{\pi/4}$ in $\mathbb{S}^n$. © 2002 Elsevier Science (USA)

Key Words: Faber–Krahn inequality; Szegö–Weinberger inequality; stability of eigenvalues; space forms; constant curvature.

0. INTRODUCTION

In the 1920s, Faber [7] and Krahn [9] proved that for all plain domains Ω with fixed volume, the ball B has the minimum first eigenvalue of the Dirichlet Laplacian

$$\lambda(B) \leq \lambda(\Omega),$$

with equality only when $\Omega \equiv B$. This physical isoperimetric inequality gives us information about how a characteristic value of the Laplacian behaves

[1] Supported by CNPq grant 150133/99-5, Brazil.

[2] Current address: Dep. de Ing. Matemática, Universidad de la Frontera Casilla 54-D, Temuco, Chile. E-mail: aavila@ufro.cl.

0022-247X/02 $35.00

if we know information about the domain (fixed volume). Moreover, if Ω is close to B in the Hausdorff distance sense, we know that $\lambda(\Omega)$ is close to $\lambda(B)$. But, can we say that Ω looks like B if $\lambda(\Omega)$ is close to $\lambda(B)$? This is the idea of *stability* that we will study here.

In [10], Melas proved that for a convex domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with fixed volume, and

$$\lambda(\Omega) \leq \lambda(B) + \epsilon,$$

with $\epsilon > 0$ small, then there exist two balls B_{r_1} and B_{r_2} such that $B_{r_1} \subset \Omega \subset B_{r_2}$ and the volume of each ball approaches the volume of Ω if $\epsilon \to 0$, that is, Ω is close to a ball in the Hausdorff distance sense. We will extend this stability result to 2-dimensional spaces of constant curvature κ. It is well known [6, Chap. 2] that we can reduce the possible cases to $\kappa = -1, 0$, and 1. For $\kappa = 1$, we denote the space by $\mathbb{S}^n$. For $\kappa = -1$, we denote the space by $\mathbb{H}^n$. We will also denote by $\mathbb{S}^n_+ = \{x \in \mathbb{S}^n : x_{n+1} \geq 0\}$ the hemisphere in $\mathbb{S}^n$. Here we denote by $|\cdot|$ the 2-dimensional Lebesgue measure on $\mathbb{S}^2$ or $\mathbb{H}^2$ and the 1-dimensional Hausdorff measure on $\mathbb{S}^2$ or $\mathbb{H}^2$.

Our first two results are the following:

THEOREM 0.1. *Let Ω be a convex domain with C^2 boundary contained in the hemisphere $\mathbb{S}^2_+$. Let $B_{\bar{\theta}}$ be the ball of the same volume as Ω. Let λ and $\bar{\lambda}$ be the first eigenvalues of the Dirichlet Laplacian on Ω and $B_{\bar{\theta}}$, respectively. Then, for $\epsilon > 0$ sufficiently small and such that*

$$\lambda \leq \bar{\lambda} + \epsilon,$$

there exist two balls B_{r_1} and B_{r_2} such that $B_{r_1} \subset \Omega \subset B_{r_2}$ and

$$|B_{r_1}| \geq (1 - C\epsilon^{\frac{1}{6}})|\Omega|, \qquad |\Omega| \geq (1 - C\epsilon^{\frac{1}{12}})|B_{r_2}|,$$

where $C > 0$ is a constant, $C = C(|\Omega|, |\partial\Omega|, \bar{\theta})$.

THEOREM 0.2. *Let Ω be a convex domain with C^2 boundary contained in $\mathbb{H}^2$. Let $B_{\bar{\theta}}$ be the ball of the same volume as Ω. Let λ and $\bar{\lambda}$ be the first eigenvalues of the Dirichlet Laplacian on Ω and $B_{\bar{\theta}}$, respectively. Then, for $\epsilon > 0$ sufficiently small and such that*

$$\lambda \leq \bar{\lambda} + \epsilon,$$

there exist two balls B_{r_1} and B_{r_2} such that $B_{r_1} \subset \Omega \subset B_{r_2}$ and

$$|B_{r_1}| \geq (1 - C\epsilon^{\frac{1}{6}})|\Omega|, \qquad |\Omega| \geq (1 - C\epsilon^{\frac{1}{12}})|B_{r_2}|,$$

where $C > 0$ is a constant, $C = C(|\Omega|, |\partial\Omega|, \bar{\theta})$.

Remark 0.1. The restriction to 2-dimensional spaces shows some lack of control in the geometry of sets on spaces of higher dimension. See our remark after the proofs.

The question about stability can be extended to other physical isoperimetric inequalities. In [18], Xu studied the stability of the Szegö–Weinberger inequality: for all bounded domains Ω with fixed volume, the ball B has the maximum first nonzero eigenvalue for the Neumann Laplacian,

$$\mu(\Omega) \leq \mu(B),$$

with equality only for the case $\Omega \equiv B$. He considered Ω contained either in $\mathbb{R}^n$ or $\mathbb{H}^n$. He extended Melas' result; that is, there exist two balls B_{r_1} and B_{r_2} such that $B_{r_1} \subset \Omega \subset B_{r_2}$ and their volumes approach the volume of Ω as $\epsilon \to 0$.

We could extend partially Xu's result to some sets on the hemisphere $\mathbb{S}^n_+$. Let $B_\alpha = \{x \in \mathbb{S}^n : 0 \leq x_{n+1} \leq \cos\alpha\}$ be the polar cap where α is the angle measured with respect to the positive side of the x_{n+1} axis. We obtained the following result:

THEOREM 0.3. *Let Ω be a convex domain contained in the polar cap $B_{\pi/4}$ and let $B_{\bar{\theta}}$ be a ball of the same volume, $|B_{\bar{\theta}}| = |\Omega|$. Let μ and $\bar{\mu}$ be the first nonzero eigenvalues of Ω and $B_{\bar{\theta}}$, respectively. Then, for $\epsilon > 0$ there exists $\delta = \delta(\epsilon, n, \bar{\theta}) > 0$ such that if $\mu > \bar{\mu} - \epsilon$ then there exist two balls $B_{\bar{\theta}-\delta}$ and $B_{\bar{\theta}+\delta}$ such that*

$$B_{\bar{\theta}-\delta} \subset \Omega \subset B_{\bar{\theta}+\delta}$$

in $\mathbb{S}^n$, and $\lim_{\epsilon\to 0} \delta = 0$.

Remark 0.2. We think that the restriction $\Omega \subset B_{\pi/4}$ can be avoided. See our remark after the proof.

In Section 2 we will prove Theorems 0.1 and 0.2 following Melas' idea of using Bonnesen-style inequalities. In Section 3 we will prove Theorem 0.3 based on Xu's work. For information about spaces of constant curvature and the Laplacian, see [6, Chap. 2 and 4].

1. STABILITY OF THE FABER–KRAHN INEQUALITY

The Faber–Krahn inequality on spaces of constant curvature on $\mathbb{S}^n$ was studied by Sperner [15] and Friedland and Hayman [8]. On $\mathbb{H}^n$, Chavel [6] has a proof for Riemannian manifolds which includes this and the previous case. To prove our theorems, we need more information about the geometry of sets in $\mathbb{S}^2$ and $\mathbb{H}^2$. Some of these facts used here are in the work of

Ashbaugh and Benguria [2], who studied the PPW inequality on $\mathbb{S}^n$. We can divide Melas' work in three main parts: first, information given by the hypothesis on the eigenvalues; second, closeness of the level sets to their inner balls through a Bonnesen-style isoperimetric inequality; and finally, relation of one of the inner balls, B_{r_1}, with Ω. About B_{r_2}, it can be obtained as in the third step.

1.1. *The Case of* $\mathbb{S}^2$

We will prove Theorem 0.1 using the Faber–Krahn proof in Chavel's book (see [6]) and an Bonnesen-style inequality in $\mathbb{S}^2$ proved by Osserman [11].

We will consider Ω **convex** in $\mathbb{S}^2$ if for any two points in Ω the geodesics connecting the two points belong to Ω.

Let $u > 0$ be the first eigenfunction of Ω such that $\int_\Omega u^2\,dw = 1$. Let $T = \max_{x\in\Omega} u(x)$; for $0 \le t \le T$ we denote

$$\Omega_t = \{x \in \Omega : u(x) > t\}, \qquad \partial\Omega_t = \{x \in \Omega : u(x) = t\}.$$

Let B_{θ_t} be the geodesic ball of radius θ_t such that $|\Omega_t| = |B_{\theta_t}|$. From the co-area formula [6, p. 86] we can define

$$V(t) \equiv |\Omega_t| = \int_t^T \int_{\partial\Omega_t} \frac{1}{|\nabla u|}\,dA\,dt,$$

where dA denotes the 1-dimensional Hausdorff measure on $\mathbb{S}^2$. Using this formulation, we can write λ in terms of the new coordinates as

$$\lambda = \int_\Omega |\nabla u|^2 = \int_0^T \int_{\partial\Omega_t} |\nabla u|\,dA\,dt. \tag{1}$$

We want to get an inequality involving λ and geometrical quantities. First, note that

$$-V'(t) = \int_{\partial\Omega_t} \frac{1}{|\nabla u|}\,dA. \tag{2}$$

Using the Cauchy–Schwarz inequality and (2),

$$|\partial\Omega_t|^2 = \left(\int_{\partial\Omega_t} \frac{|\nabla u|^{\frac{1}{2}}}{|\nabla u|^{\frac{1}{2}}}\,dA\right)^2 \le -V'(t)\int_{\partial\Omega_t} |\nabla u|\,dA. \tag{3}$$

Using (3) and the *classical isoperimetric inequality* (see [4, 12, 13])$|\partial\Omega_t| \ge |\partial B_{\theta_t}|$, we get from (1) the lower bound

$$\lambda = \int_0^T \int_{\partial\Omega_t} |\nabla u|\,dA \ge \int_0^T \frac{1}{-V'(t)}|\partial\Omega_t|^2 \ge \int_0^T \frac{1}{-V'(t)}|\partial B_{\theta_t}|^2 = \bar{\lambda}.$$

Subtracting $\bar{\lambda}$ from both sides, we get from the last inequality and the hypothesis for λ that for $0 \leq t_0 \leq T$,

$$\epsilon \geq \int_0^T \frac{1}{-V'(t)} \left(|\partial\Omega_t|^2 - |\partial B_{\theta_t}|^2 \right) \geq \int_0^{t_0} \frac{1}{-V'(t)} \left(|\partial\Omega_t|^2 - |\partial B_{\theta_t}|^2 \right) \geq 0.$$

For some $\bar{t} \in [0, t_0]$, $|\partial\Omega_t|^2 - |\partial B_{\theta_t}|^2$ attains its minimum, so our last inequality reduces to

$$\epsilon \geq \left(|\partial\Omega_{\bar{t}}|^2 - |\partial B_{\theta_{\bar{t}}}|^2 \right) \int_0^{t_0} \frac{1}{-V'(t)}. \tag{4}$$

We need to find a lower bound for $\int_0^{t_0} \frac{dt}{-V'(t)}$ with t_0 close to 0. Because Ω has a smooth boundary, it satisfies the inner ball property. Then, by using Hopf's theorem with $u > 0$ in Ω and $u = 0$ on $\partial\Omega$, we get for each point of the boundary that the normal derivative of u is negative; then $|\nabla u| > 0$ on $\partial\Omega$ [6, Section XII.11]. Using the continuity of ∇u, there exists a $\hat{t}$ such that $|\nabla u| \geq c$ for $t \leq \hat{t}$ and $c > 0$ a constant independent of t. Let us choose t_0 small enough such that

$$|\nabla u(x)| \geq c > t_0^{1/2}, \qquad x \in \Omega \backslash \Omega_{\hat{t}}. \tag{5}$$

Note that in $(0, \hat{t})$ there are only regular values and c is independent of t_0. On the other hand, notice that $|\partial\Omega_t| \leq c_\Omega$ (see our remarks).

From (5) and the definition of $V'(t)$ we get

$$-V'(t) \leq \frac{1}{t_0^{1/2}} \int_{\partial\Omega_t} dA = \frac{|\partial\Omega_t|}{t_0^{1/2}} \leq \frac{c_\Omega}{t_0^{1/2}},$$

and using this inequality in the integral of $\frac{1}{V'(t)}$ we obtain

$$\int_0^{t_0} \frac{1}{-V'(t)} \geq \frac{t_0^{3/2}}{c_\Omega}, \tag{6}$$

with t_0 near to zero. Using (6) in (4) we get

$$\epsilon > \frac{t_0^{3/2}}{c_\Omega} \left(|\partial\Omega_{\bar{t}}|^2 - |\partial B_{\theta_{\bar{t}}}|^2 \right). \tag{7}$$

In addition to this inequality, we get the bound for the difference of areas by substituting the inequality for $V'(t)$

$$|\Omega \backslash \Omega_{t_0}| = \int_0^{t_0} -V'(t)\, dt \leq c_\Omega t_0^{1/2}. \tag{8}$$

Choosing $t_0^3 = \epsilon$ in (7), we get

$$c_\Omega \epsilon^{1/2} > \left(|\partial\Omega_{\bar{t}}|^2 - |\partial B_{\theta_{\bar{t}}}|^2 \right). \tag{9}$$

Denote by θ the angle such that $|\partial\Omega_{\bar{t}}| = 2\pi \sin\theta$ if $|\partial\Omega_{\bar{t}}| < 2\pi$ and $\theta = \frac{\pi}{2}$ if $|\partial\Omega_{\bar{t}}| \geq 2\pi$. Using the fact that if $a \geq b \geq 0$, then $a^2 - b^2 \geq b(a-b)$ we find that inequality (9) changes to

$$c_\Omega \epsilon^{1/2} > 4\pi^2 \sin\left(\frac{\bar{\theta}}{2}\right)\left(\sin\theta - \sin\theta_{\bar{t}}\right), \tag{10}$$

for ϵ small enough such that $\theta_{\bar{t}} > \frac{\bar{\theta}}{2}$, where $\bar{\theta}$ is the radius of the ball of the same volume as $|\Omega|$ given in the hypothesis and $\bar{t} \in (0, \epsilon^{1/3})$.

Now we use the classical isoperimetric inequality to obtain

$$c_1\epsilon^{1/2} + |\partial B_{\theta_{\bar{t}}}| \geq |\partial\Omega_{\bar{t}}| \geq |\partial B_{\theta_{\bar{t}}}|,$$

where c_1 can be obtained from (10). Applying the mean value theorem to (10) we obtain

$$c_2\epsilon^{1/2} > \theta - \theta_{\bar{t}} > 0. \tag{11}$$

We have proved that the angle radii of $\Omega_{\bar{t}}$ and $B_{\theta_{\bar{t}}}$ are close. This implies that $\Omega_{\bar{t}}$ has not only length close to the length of $B_{\theta_{\bar{t}}}$, but also area close to the area of $B_{\theta_{\bar{t}}}$. To conclude that the geometry of Ω is close to a disk, we need information about how close is the inner circle of $\Omega_{\bar{t}}$ to Ω. First, we will compare the areas $|\Omega|$ and $|\Omega_{\bar{t}}|$ and then we will compare the areas of $B_{\theta_{\bar{t}}}$ and Ω.

For this purpose, we can obtain by using (8) that $|\Omega_t| \geq |\Omega|(1 - c_3 t^{1/2})$. In terms of $\epsilon = t_0^{1/3}$ we get

$$|\Omega_{\bar{t}}| \geq |\Omega|\left(1 - c_3\epsilon^{\frac{1}{6}}\right). \tag{12}$$

To prove that the inner ball B_r of $\Omega_{\bar{t}}$ has radius close to $\bar{\theta}$, we need a *Bonnesen-style* inequality like the one used by Melas. Osserman proved the isoperimetric inequality in $\mathbb{S}^2$ [11, Theorem 6],

$$L^2 - 4\pi A + A^2 \geq (L - \widehat{L})^2 + (A - \widehat{A})^2,$$

where L and A are the length and the area of Ω and $\widehat{L}$ and $\widehat{A}$ are the length and area of the inner ball B_r. In our case, we need an inequality as

$$|\partial\Omega_{\bar{t}}|^2 - 4\pi|\Omega_{\bar{t}}| + |\Omega_{\bar{t}}|^2 \geq (|\partial\Omega_{\bar{t}}| - |\partial B_r|)^2 \tag{13}$$

for regular values of $\bar{t}$. Because $\Omega_{\bar{t}}$ has length and area close to a ball, the right-hand side of this inequality is close to zero. Using (10), we have for $|\partial\Omega_{\bar{t}}|$, $|\partial\Omega_{\bar{t}}| \leq 2\pi\sin(\theta_{\bar{t}} + c_1\epsilon^{1/2})$, and for $|\Omega_{\bar{t}}|$, $2\pi(1 - \cos\theta_{\bar{t}}) \leq |\Omega_{\bar{t}}| \leq 2\pi(1 - \cos(\theta_{\bar{t}} + c_1\epsilon^{1/2}))$. Using these two inequalities in the left-hand side of (13) we get the upper bound

$$|\partial\Omega_{\bar{t}}|^2 - 4\pi|\Omega_{\bar{t}}| + |\Omega_{\bar{t}}|^2 \leq c_4\epsilon^{\frac{1}{2}}.$$

For the right-hand side we conclude using the mean value theorem that

$$c_4\epsilon^{\frac{1}{2}} \geq (|\partial\Omega_{\bar{t}}| - |\partial B_r|)^2 \geq c_5 \sin\frac{\bar{\theta}}{2}(\theta_{\bar{t}} - r).$$

Finally,

$$c_6\epsilon^{\frac{1}{2}} \geq \theta_{\bar{t}} - r.$$

Hence we compare $|B_r|$ and $|\Omega_{\bar{t}}|$,

$$|B_r| \geq \int_0^{\theta_{\bar{t}} - c_6\epsilon^{\frac{1}{2}}} 2\pi \sin\theta \, d\theta \geq (1 - c_7\epsilon^{\frac{1}{2}})|\Omega_{\bar{t}}|. \tag{14}$$

From (12) and (14) we get

$$|B_r| \geq (1 - c_8\epsilon^{\frac{1}{6}})|\Omega|. \tag{15}$$

Thus, we get the inclusion $B_r \subset \Omega_{\bar{t}} \subset \Omega$, with $|B_r|$ close to $|\Omega|$. We can choose the ball contained in Ω and close to it as $B_{r_1} = B_r$.

Now, we need to find a ball $B_{r_2} = B_{\bar{\theta}+h}$ for some h, including Ω with volume close to the volume of Ω.

First, consider the farthest point x^* in $\partial\Omega$ with respect to the center of B_r. Now, consider the connected component D of $\Omega \backslash B_{\bar{\theta}}$ which contains x^* and consider the geodesic triangle C with vertex at x^*, height h, base of length α, and tangent to $B_{\bar{\theta}}$.

Then, using the convexity, this geodesic triangle is contained in D. On the other hand, because $|\Omega| = |B_{\bar{\theta}}|$, it follows that

$$|B_{\bar{\theta}} \backslash \Omega| = |\Omega \backslash B_{\bar{\theta}}| \geq |C| = \alpha(1 - \cos h).$$

Notice that

$$|B_{\bar{\theta}} \backslash \Omega| \leq |B_{\bar{\theta}} \backslash B_r| \leq 2\pi(\bar{\theta} - r).$$

Let us find an upper bound for $|C|$ in terms of ϵ. Using (15) we can obtain the inequality

$$c_9\epsilon^{\frac{1}{6}} \geq \cos r - \cos\bar{\theta} \geq c_{10}(\bar{\theta} - r).$$

Now, we need to get a lower bound in terms of h. First

$$1 - \cos h \geq \frac{h^2}{2}\cos\frac{\bar{\theta}}{2}. \tag{16}$$

On the other hand, we need to show that α is bounded below by a constant. Projecting the geodesic triangle on a flat circle, we get the arc relation $r = \alpha \sin h$, and it follows that for ϵ small such that $r \geq \frac{\bar{\theta}}{2}$,

$$\alpha = \frac{r}{\sin h} \geq r \geq \frac{\bar{\theta}}{2}.$$

From the last three inequalities we finally conclude that

$$h \leq c_{11}\epsilon^{\frac{1}{12}}.$$

Then, if we consider the ball B_{r_2} with radius $\bar{\theta}+h$, all points in $\Omega\backslash B_{\bar{\theta}}$ are close to $B_{\bar{\theta}}$. Using the ideas in (14) we get

$$|B_{r_2}| \leq (1+c_{10}\epsilon^{\frac{1}{12}})|\Omega| \qquad \text{or} \qquad |\Omega| \geq (1-c_{10}\epsilon^{\frac{1}{12}})|B_{r_2}|.$$

For ϵ small, Ω satisfies the desired inequalities for volumes.

1.2. *The Hyperbolic Case*

Now we proceed to prove Theorem 0.2. The geometry behaves similarly. In this space form, the length of a circle of radius θ is given by $2\pi\sinh\theta$ and the area is given by $2\pi(\cosh\theta-1)$. Moreover, $\cosh\theta$ is increasing in contrast with the $\mathbb{S}^2$ case, where $\cos\theta$ is decreasing and its domain was restricted to angles less than $\frac{\pi}{2}$.

We can follow the proof of the first inclusion on $\mathbb{S}^2$ in Section 1.1. The main difference from the previous case is the Bonnesen-style inequality. We will use a different inequality given in Osserman [11], but equivalent to the isoperimetric inequality in $\mathbb{H}^2$. We will use

$$L^2-4\pi A-A^2 \geq \left(\frac{2\pi}{\widehat{L}}(A-\widehat{A})\right)^2.$$

We get analogous bounds for length and area by using (11):

$$|\partial\Omega_{\bar{t}}|^2-4\pi|\Omega_{\bar{t}}|-|\Omega_{\bar{t}}|^2 \leq c_1\epsilon^{\frac{1}{2}}.$$

On the other hand, for the lower bound we obtain

$$\left(\frac{2\pi}{|\partial B_r|}(|\Omega_{\bar{t}}|-|B_r|)\right)^2 \geq c_2(\theta_{\bar{t}}-r).$$

We can derive the inequality

$$c_3\epsilon^{\frac{1}{2}} \geq \theta_{\bar{t}}-r.$$

Following an analogous inequality as (14) we get

$$|B_r| \geq (1-c_4\epsilon^{\frac{1}{6}})|\Omega|.$$

So we can conclude that $B_r \subset \Omega_t \subset \Omega$ and $B_{r_1}=B_r$.

For the exterior ball, we just need to change $\sin\theta$ to $\sinh\theta$ and $1-\cos\theta$ to $\cosh\theta-1$; we conclude the existence of B_{r_2},

$$|\Omega| \geq (1-c_{10}\epsilon^{\frac{1}{12}})|B_{r_2}|.$$

With this last inequality we have proved the stability case for $\mathbb{H}^2$.

Remark 1.1. Using the convexity of the level sets for $\mathbb{R}^n$ (see [3, 5]) and in $\mathbb{S}^n$ [14], we have that $c_\Omega \leq |\partial\Omega|$. Unfortunately, in $\mathbb{H}^n$ there is no convexity of the level sets (see [14]), so we must bound $c_\Omega \leq \max_t |\partial\Omega_t|$, or by continuity of $|\partial\Omega_t|$ we can bound by $|\partial\Omega| + \tilde{\epsilon}$ for t close to zero.

Remark 1.2. The power for ϵ in (15) is given by choosing $t_0^3 = \epsilon$. For the exterior ball, the power is a square root given by the geometric behavior of the area on $\mathbb{S}^2$ in (16). If we keep the constant c in (5) as a lower bound and consider $t_0^2 = \epsilon$ we can improve our exponent to $\frac{1}{2}$ instead of $\frac{1}{6}$ in the bounds of volumes given in our theorem. The problem is that our constant C gets worse. Moreover, we can still leave t_0 independent of ϵ, and we can improve the exponent to 1.

Remark 1.3. We could not extend the results for $n > 2$ because of lack of knowledge of Bonnesen-style isoperimetric inequalities for spaces of constant curvature in higher dimensions. The key point in stability is to know how we can relate the volumes of the set and an inner ball, that is, a Bonnesen-style inequality.

2. STABILITY OF THE SZEGÖ–WEINBERGER INEQUALITY

The proof of the Szegö–Weinberger inequality [16, 17] on $\mathbb{S}^n$ was done by Ashbaugh and Benguria [1]. This work has very helpful information about the growth of the functions involved in the numerator and denominator of the Rayleigh–Ritz characterization of μ. Our proof uses several results given in this work. We divide the proof of Theorem 0.3 in three steps.

2.1. *The Rayleigh-Ritz Formula*

We will first use a trial function related to the solution to the radial problem and then, using the Rayleigh–Ritz characterization of eigenvalues, we will find bounds for μ in terms of $\bar{\mu}$.

Let g be the solution to

$$-\frac{1}{\sin^{n-1}\theta}\frac{d}{d\theta}\left(\sin^{n-1}\theta\frac{dg}{d\theta}\right) + \frac{n-1}{\sin^2\theta}g = \bar{\mu}g,$$
$$g(0) \text{ finite}, \;\; g'(\bar{\theta}) = 0.$$

In $\mathbb{S}^n$, we define

$$G(\theta) = \begin{cases} g(\theta) & 0 \leq \theta \leq \bar{\theta}, \\ g(\bar{\theta}) & \bar{\theta} \leq \theta \leq \frac{\pi}{2} \end{cases}$$

and then extend by reflection about $\frac{\pi}{2}$ to $[0, \pi]$.

Consider the set of functions $G(\theta)u_i(\theta), i=1\cdots n$ with $u_i(\theta)=x_i/\sin\theta \in \mathbb{S}^n$. For $i=1\cdots n, \{x_i\}_{i=1\cdots n+1}$ is a Cartesian coordinate system with θ representing the angle from the positive side of the axis x_{n+1}. Defining $\widetilde{G}(\theta) = \frac{G(\theta)}{\sin\theta}$ and using the *center of mass* argument in Theorem 2.1 [1], we can choose $\{x_i\}_{i=1,\dots,n}$ such that the integrals on Ω are zero; that is,

$$\int_\Omega G(\theta)u_i(\theta) = 0, \qquad \text{for } i = 1\cdots n.$$

Using each function $f_i = Gu_i$ on the Rayleigh–Ritz characterization of μ and then summing up we get

$$\mu \leq \frac{\int_\Omega \Gamma(\theta)}{\int_\Omega G^2(\theta)}, \tag{17}$$

where

$$\Gamma(\theta) = \begin{cases} \left(\frac{dg}{d\theta}\right)^2 + (n-1)\frac{g^2}{\sin^2\theta} & 0 \leq \theta \leq \bar{\theta}, \\ (n-1)\frac{g^2(\bar{\theta})}{\sin^2\theta} & \bar{\theta} \leq \theta \leq \frac{\pi}{2}. \end{cases}$$

By Lemma 3.2 in [1], it was proved that g is strictly increasing on $[0, \bar{\theta}]$ if $\bar{\theta} \leq \frac{\pi}{2}$. By assumption, we can choose g positive in $(0, \bar{\theta}]$. By Theorem 4.1 in [1], we get that Γ is decreasing and G is increasing on $[0, \frac{\pi}{2}]$.

Because Ω is contained in a hemisphere, we need to *move* our integrals to the hemisphere centered at the center of mass (which we have placed at the north pole). Let $\Omega_{\pm} = \{x \in \Omega : \pm x_{n+1} > 0\}$. Notice that since $-\Omega$ is contained in a different hemisphere than Ω, $-\Omega_- \cap \Omega_+ = \emptyset$. Let us define $\widetilde{\Omega} = \Omega_+ \cup (-\Omega_-)$, which is contained in the hemisphere centered at the center of mass.

Because of the definition of G, the integrals over Ω have the same value as the integrals over $\widetilde{\Omega}$, so we can consider on $\mathbb{S}^n_+$

$$\mu \leq \frac{\int_{\widetilde{\Omega}} \Gamma(\theta)}{\int_{\widetilde{\Omega}} G^2(\theta)}. \tag{18}$$

Moreover, note that (17) and (18) are equal because G and Γ are symmetric with respect to $\frac{\pi}{2}$.

2.2. *Inclusion* $\Omega \subset B_{\bar{\theta}+\delta}$

Let us get an upper bound for the numerator and a lower bound for the denominator with respect to the integrals on the ball $B_{\bar{\theta}}$ of the same volume as Ω.

We study the denominator and the numerator separately. From the denominator in (17) we split $\Omega = (B_{\bar\theta} \cup \Omega \backslash B_{\bar\theta}) \backslash (B_{\bar\theta} \backslash \Omega)$,

$$\int_\Omega g^2(\theta) = \int_{B_{\bar\theta}} g^2(\theta) + \int_{\Omega\backslash B_{\bar\theta}} g^2(\theta) - \int_{B_{\bar\theta}\backslash\Omega} g^2(\theta).$$

Since $|\Omega| = |B_{\bar\theta}|$, we get that $|\Omega\backslash B_{\bar\theta}| = |B_{\bar\theta}\backslash\Omega|$. From Section 3.1 we know that $G(\theta)$ is nondecreasing on $[0, \frac{\pi}{2}]$, then the sum of the last two integrals is positive, which implies that

$$\int_\Omega g^2(\theta) \geq \int_{B_{\bar\theta}} g^2(\theta). \tag{19}$$

For the numerator, we consider a ball $B_{\bar\theta+r_1}$ containing Ω centered at our origin such that it is tangent at least to one point of $\partial\Omega$. Let us call one of these points x^+. Notice that our condition $\Omega \subset B_{\pi/4}$ implies that $\bar\theta + r_1 < \frac{\pi}{2}$. Now, let us split the integral as we just did in the numerator

$$\int_\Omega \Gamma(\theta) = \int_{B_{\bar\theta}} \Gamma(\theta) + \int_{\Omega\backslash B_{\bar\theta}} \Gamma(\theta) - \int_{B_{\bar\theta}\backslash\Omega} \Gamma(\theta). \tag{20}$$

For our purposes, we want to get a negative bound for the last two integrals. We will refer to an intermediate value for the radius to get that condition. Let us split

$$\Omega\backslash B_{\bar\theta} = M_1 \cup M_2,$$

where $M_1 = \Omega\backslash B_{\bar\theta+\epsilon_1}$ and $M_2 = \Omega \cap (B_{\bar\theta+\epsilon_1}\backslash B_{\bar\theta})$ with $0 < \epsilon_1 < r_1$. Note that $\Gamma(\theta) = ((n-1)g(\bar\theta)^2)/\sin^2\theta$ if $\theta > \bar\theta$ and is decreasing. We get

$$\int_{\Omega\backslash B_{\bar\theta}} \Gamma(\theta) \leq (n-1)g(\bar\theta)^2\left(\frac{|M_1|}{\sin(\bar\theta+\epsilon_1)^2} + \frac{|M_2|}{\sin^2\bar\theta}\right).$$

Adding and subtracting $|M_1|\Gamma(\bar\theta)$ we get

$$\int_{\Omega\backslash B_{\bar\theta}} \Gamma(\theta) \leq (n-1)g(\bar\theta)^2\left(\frac{|M_1|}{\sin(\bar\theta+\epsilon_1)^2} - \frac{|M_1|}{\sin^2\bar\theta}\right) + \int_{\Omega\backslash B_{\bar\theta}} \Gamma(\bar\theta).$$

Now our bound for the numerator is

$$\int_\Omega \Gamma(\theta) \leq \int_{B_{\bar\theta}} \Gamma(\theta) + (n-1)g(\bar\theta)^2|M_1|\left(\frac{1}{\sin(\bar\theta+\epsilon_1)^2} - \frac{1}{\sin^2\bar\theta}\right)$$
$$+ \int_{\Omega\backslash B_{\bar\theta}} \Gamma(\bar\theta) - \int_{B_{\bar\theta}\backslash\Omega} \Gamma(\theta).$$

Because $\Gamma(\theta)$ is decreasing, the sum of last two integrals is less than zero. Using the fact that $\sin\theta$ is increasing, we finally get the upper bound

$$\int_\Omega \Gamma(\theta) \leq \int_{B_{\bar\theta}} \Gamma(\theta) - (n-1)g(\bar\theta)^2|M_1|\left(\frac{1}{\sin^2\bar\theta} - \frac{1}{\sin(\bar\theta+\epsilon_1)^2}\right). \tag{21}$$

Using (17), (19), and (21) we get

$$\mu < \bar{\mu} - c_0|M_1|\left(\frac{1}{\sin^2\bar{\theta}} - \frac{1}{\sin(\bar{\theta}+\epsilon_1)^2}\right).$$

By our hypothesis $\bar{\mu} - \epsilon < \mu$ we can conclude that

$$\epsilon > c_0|M_1|\left(\frac{1}{\sin^2\bar{\theta}} - \frac{1}{\sin(\bar{\theta}+\epsilon_1)^2}\right) > 0, \tag{22}$$

for $0 < \epsilon_1 < r_1$.

Notice that if ϵ_1 increases, then $|M_1|$ decreases and $1/\sin^2\bar{\theta} - 1/(\sin(\bar{\theta}+\epsilon_1)^2)$ increases. If ϵ_1 decreases, then $|M_1|$ increases and $1/\sin^2\bar{\theta} - 1/(\sin(\bar{\theta}+\epsilon_1)^2)$ decreases. So we can guess that the right-hand side of (22) has an upper bound depending on r_1. Moreover, we can control r_1 with ϵ, which is our goal. Using the same idea as in Section 1, we can inscribe a cone $C \subset M_1$ with vertex at x^+ and angle γ and tangent to $B_{\bar{\theta}+\epsilon_1}$. Again, the convexity of Ω is needed to ensure the existence of such a cone. Let us get a lower bound for the volume of C in $\mathbb{S}^n$,

$$\begin{aligned}|M_1| \geq |C| &= c_1\int_0^{r_1-\epsilon_1}\int_0^{\frac{\theta\gamma}{2}} \sin^{n-2} t\, dt\, d\theta,\\ &> c_2\int_0^{r_1-\epsilon_1} \theta^{n-1} d\theta = c_3(r_1-\epsilon_1)^n,\end{aligned}$$

where c_3 is a positive constant and it depends only on n and γ.

Now, let us study the difference of sines. Using the mean value theorem and $(\cos\theta)' < 0$, we obtain the lower bound

$$\frac{1}{\sin^2\bar{\theta}} - \frac{1}{\sin(\bar{\theta}+\epsilon_1)^2} \geq c_4\epsilon_1,$$

where c_4 involves a term $\cos\hat{\theta}$ with $\bar{\theta}+r_1 \leq \hat{\theta} < \frac{\pi}{2}$, $\hat{\theta}$ fixed. Finally, using the last two inequalities in (22) we get

$$\epsilon > c_5\epsilon_1(r_1-\epsilon_1)^n.$$

Considering that $0 < \epsilon_1 < r_1$ we maximize for ϵ_1 the right-hand side and replace $\epsilon_1^* = r_1/(n+1)$ in the last inequality to get

$$\epsilon > c_6 r_1^{n+1}. \tag{23}$$

We conclude that r_1 must be small and it goes to zero if ϵ goes to zero. Notice that c_6 depends on n, $\bar{\theta}$, and $\hat{\theta}$.

2.3. *Second Inclusion*

Now let us consider $B_{\bar{\theta}-r_2}$ to be the ball centered at our origin contained in Ω such that it is tangent at least to one point x^- in $\partial\Omega$. We will use the same argument as that in the last section, but we will need to be careful about the use of the properties of $g(\theta)$ and $\Gamma(\theta)$. In this case $\Gamma(\theta)$ is bigger in $B_{\bar{\theta}}\backslash\Omega$ than in $\Omega\backslash B_{\bar{\theta}}$; then we will fix our calculations on $B_{\bar{\theta}}\backslash\Omega$.

We will find a lower bound for the last integral in (20). Let us split $B_{\bar{\theta}}\backslash\Omega = N_1 \cup N_2$, with $N_1 = (B_{\bar{\theta}}\backslash B_{\bar{\theta}-\epsilon_2})\backslash\Omega$ and $N_2 = B_{\bar{\theta}-\epsilon_2}\backslash\Omega$ for $0 < \epsilon_2 < r_2$. Because $\Gamma(\theta)$ is decreasing we get $\Gamma(\theta) > \Gamma(\bar{\theta})$ in N_1 and $\Gamma(\theta) > \Gamma(\bar{\theta} - \epsilon_2)$ in N_2. Then

$$\int_{B_{\bar{\theta}}\backslash\Omega} \Gamma(\theta) > \int_{N_1} \Gamma(\bar{\theta}) + \frac{(n-1)|N_2| g^2(\bar{\theta}-\epsilon_2)}{\sin^2(\bar{\theta}-\epsilon_2)}.$$

Adding and subtracting $|N_2|\Gamma(\bar{\theta})$ we get

$$\int_{B_{\bar{\theta}}\backslash\Omega} \Gamma(\theta) > \int_{B_{\bar{\theta}}\backslash\Omega} \Gamma(\bar{\theta}) + (n-1)|N_2| \left(\frac{g^2(\bar{\theta}-\epsilon_2)}{\sin^2(\bar{\theta}-\epsilon_2)} - \frac{g^2(\bar{\theta})}{\sin^2\bar{\theta}} \right).$$

Then

$$\mu \int_{B_{\bar{\theta}}} g^2(\theta) < \int_{B_{\bar{\theta}}} \Gamma(\theta) - (n-1) \int_{N_2} \left(\frac{g^2(\bar{\theta}-\epsilon_2)}{\sin^2(\bar{\theta}-\epsilon_2)} - \frac{g^2(\bar{\theta})}{\sin^2\bar{\theta}} \right)$$
$$+ \int_{\Omega/B_{\bar{\theta}}} \Gamma(\theta) - \int_{B_{\bar{\theta}}\backslash\Omega} \Gamma(\bar{\theta}).$$

Because Γ is nonincreasing, the sum of the last two integrals of Γ is negative and, thus, using the hypothesis on $\bar{\mu}$, we obtain

$$\epsilon > c_7 |N_2| \left(\frac{g^2(\bar{\theta}-\epsilon_2)}{\sin^2(\bar{\theta}-\epsilon_2)} - \frac{g^2(\bar{\theta})}{\sin^2\bar{\theta}} \right).$$

The difference involving $g^2(\theta)/\sin^2\theta$ is the different term here compared to (22). Following the same analysis as in the previous section, we bound the volume of N_2 by another cone $C \subset N_2$ of angle γ small and vertex in $x^- \in \partial\Omega \cap B_{\bar{\theta}-\epsilon_2}$. We get a bound for the volume

$$|N_2| \geq |C| = c_8 (r_2 - \epsilon_2)^n.$$

For the difference between sines we use again the mean value theorem. We need to know whether $\frac{g(\theta)}{\sin\theta}$ is decreasing or not in $(\bar{\theta} - \epsilon_2, \bar{\theta})$. Using the auxiliar function $q(\theta) = \sin\theta \frac{g'(\theta)}{g(\theta)}$ given in [1] and using the property that $q < \cos\theta$ [1, Theorem 4.1], we notice that

$$\left(\frac{g(\theta)}{\sin\theta} \right)' = \frac{g(\theta)}{\sin^2\theta} (q(\theta) - \cos\theta) < 0$$

in $(0, \bar{\theta})$. Moreover, q' is negative. We can find a lower bound such that

$$\frac{g^2(\bar{\theta}-\epsilon_2)}{\sin^2(\bar{\theta}-\epsilon_2)} - \frac{g^2(\bar{\theta})}{\sin^2\bar{\theta}} \geq c_9\epsilon_2,$$

with c_9 depending on a term $g(\hat{\theta})(\cos\hat{\theta} - q(\hat{\theta}))$ with $0 < \hat{\theta} \leq \bar{\theta} - r_2$ fixed. The value $\hat{\theta}$ is positive because Ω is convex, and then it contains the *center of mass*. Thus we can find the inequality for ϵ_2,

$$\epsilon > c_{10}\epsilon_2(r_2-\epsilon_2)^n, \qquad \epsilon_2 \in (0, r_2).$$

We can choose $\epsilon_2 = r_2/(n+1)$ and then $\epsilon > c_{11}r_2^{n+1}$.

We get a result similar to (23) to conclude a bound for the size of r_2; that is, r_2 must be small. The constant c_{11} depends on n, $\hat{\theta}$, and $\bar{\theta}$.

Choosing $\delta = \max\{r_1, r_2\}$ we can conclude our result in the theorem.

Remark 2.1. We notice that our power for ϵ is better than the one given by Melas. We notice that in his case, the choice of $t_0^2 = \epsilon$ gave him the factor $\frac{1}{2n}$.

Remark 2.2. Notice that convexity is not a necessary condition. The key point is to get an upper bound of $|M_1|$ which goes to zero as $\epsilon_1 \to r_1$. Moreover, we can think about small *non-convex* perturbation of the surface of the ball, keeping volume and surface area small.

Remark 2.3. We think the restriction $\Omega \subset B_{\pi/4}$ can be avoided. Melas [10] proved the stability for the ratio of the first two eigenvalues of the Dirichlet Laplacian, that is, the stability of the Payne–Pólya–Weinberger inequality on $\mathbb{R}^n$. This proof uses a framework similar to the proof of the Szegö-Weinberger inequality on $\mathbb{S}^n$. So, we expect that if we can extend the stability result for the ratio of the first two eigenvalues on $\mathbb{S}^n_+$, then we will have some idea about how we can extend the stability result of μ to $\mathbb{S}^n_+$.

ACKNOWLEDGMENT

I express my sincere thanks to Professor M. S. Ashbaugh for giving me this problem as a thesis work and for supporting me with his friendly collaboration.

REFERENCES

1. M. S. Ashbaugh and R. D. Benguria, Sharp upper bounds to the first nonzero Neumann eigenvalue for bounded domains in spaces of constant curvature, *J. London Math. Soc.* **52** (1995), 402–416.

2. M. S. Ashbaugh and R. D. Benguria, A sharp bound for the ratio of the first two Dirichlet eigenvalues of a domain in a hemisphere of $\mathbb{S}^n$, *Trans. Amer. Math. Soc.* **353** (2001), 1055–1087.
3. H. J. Brascamp and E. H. Lieb, On extensions of the Brunn–Minkowski and Prékopa–Leindler theorems including inequalities for log concave functions and with an application to a diffusion equation, *J. Funct. Anal.* **22** (1976), 366–389.
4. Yu. D. Burago and V. A. Zalgaller, "Geometric Inequalities," Grundlehren der mathematischen Wissenschaften, Vol. 285, Springer-Verlag, Berlin, 1988.
5. L. A. Caffarelli and J. Spruck, Convexity properties of solutions to some classical variational problems, *Comm. Partial Differential Equations* **7** (1982), 1337–1379.
6. I. Chavel, "Eigenvalues in Riemannian Geometry," Series in Pure and Applied Mathematics, Vol. 115, Academic Press, Orlando, FL, 1984.
7. G. Faber, Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt, *in* "Sitzungberichte der mathematisch-physikalischen Klasse der Bayerischen Akademie der Wissenschaften zu München Jahrgang," pp. 169–172. BADW, Munich, 1923.
8. S. Friedland and W. K. Hayman, Eigenvalue inequalities for the Dirichlet problem on spheres and the growth of subharmonic functions, *Comment. Math. Helv.* **51** (1976), 133–161.
9. E. Krahn, Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises, *Math. Ann.* **94** (1925), 97–100.
10. A. Melas, The stability of some eigenvalue estimates, *J. Differential Geom.* **36** (1992), 19–33.
11. R. Osserman, Bonnesen-style isoperimetric inequalities, *Amer. Math. Monthly* **86** (1979), 1–29.
12. E. Schmidt, Die Brunn–Minkowski Ungleichung und ihr Spiegelbild sowie die isoperimetrische Eigenschaft der Kugel in der Euklidische und nichteuklidischen Geometrie, I, *Math. Nachr.* **1** (1948), 81–157.
13. E. Schmidt, Die Brunn–Minkowski Ungleichung und ihr Spiegelbild sowie die isoperimetrische Eigenschaft der Kugel in der Euklidische und nichteuklidischen Geometrie, II, *Math. Nachr.* **2** (1949), 171–244.
14. Y. Shih, A counterexample to the convexity property of the first eigenfunction on a convex domain of negative curvature, *Comm. Partial Differential Equations* **14** (1989), 867–876.
15. E. Sperner, Zur Symmetrisierung von Funktionen auf Sphären, *Math. Z.* **134** (1973), 317–327.
16. G. Szegö, Inequalities for certain eigenvalues of a membrane of given area, *J. Rational Mech. Anal.* **3** (1954), 343–356.
17. H. F. Weinberger, An isoperimetric inequality for the n-dimensional free membrane problem, *J. Rational Mech. Anal.* **5** (1956), 633–636.
18. Y. Xu, The first nonzero eigenvalue of Neumann problem on Riemannian manifolds, *J. Geom. Anal.* **5** (1995), 151–165.