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Étale double covers of cyclic p-gonal covers ☆

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This paper computes the Galois group of the Galois cover of the composition of an étale double cover of a cyclic p-gonal cover for any prime p. Moreover a relation between some of its Prym varieties and the Jacobian of a subcover is given. In a sense this generalizes the trigonal construction.

1. Introduction

In this paper we investigate the Galois group of the Galois cover of the composition of étale double coverings $Y \to X$ of cyclic covers $X \to \mathbb{P}^1$ of prime degree p. For $p = 2$, Mumford shows in [6] that $Y \to \mathbb{P}^1$ is Galois with Galois group the Klein group of order 4 and the Prym variety $P(Y/X)$ is isomorphic as a principally polarized abelian variety to either a Jacobian or the product of 2 Jacobians.

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For $p = 3$, the trigonal construction tells us the principally polarized $P(Y/X)$ is isomorphic to a Jacobian of a tetragonal curve. In Section 2 we study the Galois group of the Galois closure $Z \to \mathbb{P}^1$ of $Y \to \mathbb{P}^1$ for an odd prime p. The main result of this section is

Theorem 2.6. *Let p be an odd prime, $Y \to X$ an étale double cover and $X \to \mathbb{P}^1$ a cyclic cover of degree p. Then $Y \to \mathbb{P}^1$ is not Galois. Denoting by $Z \to \mathbb{P}^1$ its Galois closure, its Galois group G is*

$$G = N \rtimes P$$

where $N \cong \mathbb{Z}_2^{p-1}$ and $P \cong \mathbb{Z}_p$, and $X = Z/N$, $Y = Z/H$, with H a maximal subgroup of N.

There are $2^{p-1} - 1$ maximal subgroups of N. The group P acts on them by conjugation and there are $m := \frac{1}{p}(2^{p-1} - 1)$ conjugacy classes of such subgroups.
Let $\{Y_i \to X \mid i = 1, \ldots, m\}$ be the corresponding double covers. It is easy to see that they are all étale. If $T := Z/P$, there is a natural homomorphism

$$\alpha : \prod_{i=1}^{m} P(Y_i/X) \to JT.$$

Our main result is:

Theorem 3.1. $\alpha : \prod_{i=1}^{m} P(Y_i/X) \to JT$ *is an isogeny with kernel in the 2^{p-2}-division points.*

As an immediate consequence we get examples of Jacobians with arbitrarily many isogeny factors of the same dimension. For $p = 3$ this is not yet the trigonal construction, which, however, is an easy consequence, as we show in Section 4.

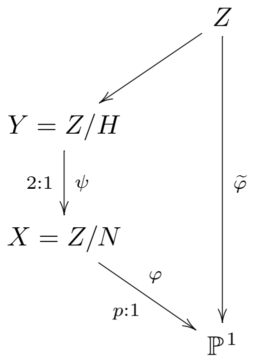
For the sake of completeness, we also consider the case $p = 2$, i.e., we give a proof of Mumford's theorem mentioned above. Note that Mumford gives only a short sketch of proof leaving the details to the reader. It seems to us that our proof is different from the one Mumford had in mind.

2. Étale covers of cyclic p-gonal covers

2.1. The structure of the Galois group

Let p be a prime and $\varphi : X \to \mathbb{P}^1$ be a cyclic covering of degree p ramified over β points of $\mathbb{P}^1$, with $\beta \geq 3$. Observe that if $p = 2$ then β must be even.

Let $\psi : Y \to X$ be an étale double cover and $\widetilde{\varphi} : Z \to \mathbb{P}^1$ the Galois closure of the composed map $\varphi \circ \psi$. Let G denote the Galois group of $\widetilde{\varphi}$ and H and N the subgroups of G corresponding to Y and X. So we have the following commutative diagram



(2.1)

In this section we determine the structure of G.

Lemma 2.1. *The permutation representation ρ of G on the right cosets of H in G has its image in the alternating group A_{2p} of degree $2p$, and the non-trivial elements of G fixing points in Z have order p. Moreover, the representation $\rho : G \to A_{2p}$ is injective.*

Proof. Recall that $Y \to X$ is the double covering corresponding to the embedding $H \subset N$. Since φ is cyclic of prime degree, the local monodromy of each of its branch points is a cycle of length p. Since ψ is an étale double cover, every local monodromy of $\varphi \circ \psi$ above a branch point is the product of two disjoint cycles of length p and hence in A_{2p}. Since G is generated by these products, this gives the first assertion. The second assertion is clear. □

Corollary 2.2. *If $p = 2$, the covering $\varphi \circ \psi$ is Galois with Galois group G the Klein group of order 4. In particular $Z = Y$.*

Proof. According to Lemma 2.1, G is a subgroup of A_4, generated by elements which are products of two disjoint cycles of length 2. Hence G is the Klein group of order four. □

Proposition 2.3. *Suppose p is an odd prime. Then the covering $\varphi \circ \psi : Y \to \mathbb{P}^1$ cannot be Galois.*

Proof. Since groups of order $2p$ cannot be generated by elements of order p, the covering $\varphi \circ \psi : Y \to \mathbb{P}^1$ cannot be Galois. □

For the rest of this section we assume that p is an odd prime; hence the covering $Y \to \mathbb{P}^1$ is not Galois, so $Z \neq Y$ and H and N are the proper subgroups of G corresponding to Y and X respectively, as in Diagram (2.1).

Let $\{1 = g_1, g_2, \dots, g_p\}$ denote a complete set of representatives of right cosets of N in G and $\{1 = n_1, n_2\}$ denote a complete set of representatives of right cosets of H in N. Then the set $\{n_i g_j : i = 1, \dots, p, j = 1, 2\}$ is a complete set of representatives of right cosets of H in G.

For $i = 1, \dots, p$ consider

$$\Delta_i := \{Hn_1 g_i, Hn_2 g_i\}$$

as a set of two elements. Then the right action of G on the right cosets of H in G induces a transitive action of G on the set

$$\Omega := \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_p.$$

This is just the right action of G on the right cosets of N in G. Now denote for $i = 1, \dots, p$,

$$H_i := g_i^{-1} H g_i. \tag{2.2}$$

Clearly each H_i is a normal subgroup of index 2 in N.

Lemma 2.4.

(i) *Any element of H_i stabilizes each of the two points of Δ_i;*
(ii) *N is the stabilizer of each set Δ_i.*

Proof. For (i) use that H is normal in N. By definition, N is the normal subgroup of G corresponding to the covering $X \to \mathbb{P}^1$. Since the Δ_i represent a right coset of N in G, this implies (ii). One can also see this directly: suppose $n \in N$. For any $i, 1 \le i \le p$ there is an $n_i' \in N$ such that $n = g_i^{-1} n_i' g_i$. Then we have

$$\Delta_i n = \{Hn_1 g_i, Hn_2 g_i\} g_i^{-1} n_i' g_i = \{Hn_1 n_i' g_i, Hn_2 n_i' g_i\} = \Delta_i.$$

Since G does not stabilize Δ_i and N has prime index in G, we conclude that N is the stabilizer of Δ_i. $\square$

Recall the representation $\rho : G \to A_{2p}$. Since N is a normal subgroup of index p in G, we may enumerate the right cosets Δ_i of N in G in such a way that we can identify the set Δ_i with the set $\{i, p+i\}$ and the action of G on the Δ_i corresponds to the permutation (right-)action of the group A_{2p} on the set $\{1, \dots, 2p\}$. Moreover, fixing a branch point, we may enumerate its branches in such a way that the local monodromy around this point is given by the cycle

$$\sigma := (1, 2, \dots, p)(p+1, p+2, \dots, 2p).$$

Lemma 2.5.

$$N \cong (\mathbb{Z}_2)^{p-1}.$$

Proof. Consider, for $i = 1, \dots, p$, the transposition $t_i := (i, p+i)$. Certainly t_i is not contained in G, since $G \subset A_{2p}$. However, we have

$$s_1 := t_1 t_2 \in N,$$

since it stabilizes each set $\{(i, p+i)\}$ and so is in N by Lemma 2.4 and the identifications. Moreover,

$$\begin{aligned} \sigma^{-1} t_1 t_2 \sigma &= t_2 t_3 =: s_2 \in N \\ \sigma^{-2} t_1 t_2 \sigma^2 &= t_3 t_4 =: s_3 \in N \\ &\cdots \\ &\cdots \\ \sigma^{-(p-1)} t_1 t_2 \sigma^{p-1} &= t_p t_1 =: s_p \in N \end{aligned} \tag{2.3}$$

which gives

$$\prod_{i=1}^{p} s_i = (t_1 t_2)(\sigma^{-1} t_1 t_2 \sigma)(\sigma^{-2} t_1 t_2 \sigma^2) \cdots (\sigma^{-(p-1)} t_1 t_2 \sigma^{p-1}) = 1.$$

Since the cycles s_i pairwise commute, and clearly there is no non-trivial relation between the cycles $s_1, s_2, \dots s_{p-1}$, this implies

$$2^{p-1} = |\langle s_1, s_2, \dots, s_p \rangle| \leq |N|.$$

Since a non-trivial element of $H_1 \cap H_2 \cap \ldots \cap H_{p-1}$ would be the transposition exchanging the two points of Δ_p, which is not in A_{2p}, we have $H_1 \cap H_2 \cap \ldots \cap H_{p-1} = \{1\}$. Consider the group homomorphism

$$\Phi' : N \to N/H_1 \times N/H_2 \times \ldots \times N/H_{p-1}$$

defined by $\Phi'(n) = (H_1 n, H_2 n, \dots, H_{p-1} n)$. Since $\ker(\Phi') = \bigcap_{i=1}^{p-1} H_i = \{1\}$, we have $N \cong N/\ker \Phi' \lesssim (\mathbb{Z}_2)^{p-1}$. Hence $N \cong (\mathbb{Z}_2)^{p-1}$. □

Theorem 2.6. *Let $X \to \mathbb{P}^1$ be a cyclic covering of degree an odd prime p, and $Y \to X$ be an étale double covering. Let $Z \to \mathbb{P}^1$ be the Galois closure of the composition $Y \to \mathbb{P}^1$ with Galois group G. With the notation of above, if P denotes the subgroup of G generated by the cycle σ, then G is the semi-direct product*

$$G = N \rtimes P \simeq \mathbb{Z}_2^{p-1} \rtimes \mathbb{Z}_p.$$

Proof. Since N is a normal subgroup of index p in G and $|N| = 2^{p-1}$ we have $G = N \rtimes P$. $\square$

A presentation of G is given as

$$G = \langle s_1, \dots, s_p, \sigma \mid \prod_{i=1}^{p} s_i = 1, \sigma^p = 1,\ s_1^2 = 1,\ \sigma^{-1} s_j \sigma = s_{j+1} \text{ for } j = 1, \dots, p-1 \rangle. \tag{2.4}$$

2.2. The subcovers of Z

Let p denote an odd prime. According to a well-known result of elementary number theory, the number

$$m := \frac{1}{p}(2^{p-1} - 1) \tag{2.5}$$

is an integer. The abelian group N has exactly m conjugacy classes of maximal subgroups with respect to the action of P. For each $1 \le j \le m$ consider R_j a representative of the corresponding conjugacy class of maximal subgroups of N. Here $R_1 = H$.

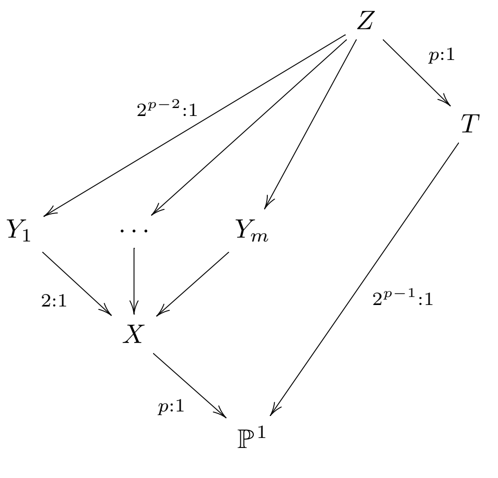
To each subgroup R_j corresponds a double covering of X. Let

$$Y_j := Z/R_j \quad \text{for} \quad j = 1, \dots, m$$

denote the corresponding curves. In particular, $Y_1 = Y$.

Denoting moreover $T := Z/P$, we have the following diagram

(2.6)



Lemma 2.7. *The map $Z \to X$ of degree 2^{p-1} is étale. In particular, all covers $Y_i \to X$ of the above diagram are étale.*

Proof. This follows immediately from Lemma 2.1. □

Proposition 2.8. *Let β denote the number of branch values of $X \to \mathbb{P}^1$, with $\beta \geq 3$. Then the genera of the curves in the above diagram are:*

- $g(X) = \frac{p-1}{2}(\beta - 2)$;
- $g(Y_i) = (p-1)(\beta - 2) - 1$;
- $g(Z) = 2^{p-2}(p-1)\beta - (p2^{p-1} - 1)$;
- $g(T) = \frac{2^{p-1}-1}{p}(\frac{p-1}{2}\beta - p)$.

Proof. The first 3 assertions are obvious, since $X \to \mathbb{P}^1$ is totally ramified and $Z \to X$ is étale. For the last assertion note that over each branch value of $T \to \mathbb{P}^1$ there are $m = \frac{2^{p-1}-1}{p}$ branch points of index $p-1$ and one point étale over $\mathbb{P}^1$. So the Hurwitz formula gives the assertion. □

As an immediate consequence we obtain the following result.

Corollary 2.9. *If $P(Y_i/X)$ denotes the Prym variety of Y_i/X, we have*

$$\sum_{i=1}^{m} \dim P(Y_i/X) = \dim JT.$$

This suggests that there is a relation between the Prym varieties $P(Y_i/X)$ and the Jacobian JT. The aim of this paper is to study the relation.

2.3. The rational representations of G

We follow [9, Section 8.2] to determine the irreducible representations of a semidirect product $G = N \rtimes P$. Let $\widehat{N}$ be the character group of N. The group P acts on $\widehat{N}$ in the usual way. The stabilizer in P of the trivial character χ_0 of N is P itself, whereas the stabilizer of any non-trivial complex irreducible character of N consists of $\{1\}$ only. Hence there are exactly $1 + m$ orbits of the action of P on $\widehat{N}$, with m as in (2.5). Let $\chi_0, \chi_1, \dots, \chi_m$ be a system of representatives of these orbits, $\rho_0, \dots, \rho_{p-1}$ (ρ_0 the trivial representation) the irreducible representations of the cyclic group P and η the trivial character of $\{1\}$.

According to [9, Proposition 25]

$$\{\chi_0 \otimes \rho_j, \;\; \mathrm{Ind}_N^G(\chi_i \otimes \eta) \;\; / \;\; 0 \leq j \leq p-1, \;\; 1 \leq i \leq m\}$$

is the set of all complex irreducible representations of G. The next result follows immediately.

Corollary 2.10. *The rational irreducible representations of G are exactly the trivial representation $\rho_0 = \chi_0 \otimes \rho_0$, the representations $\theta_i = \mathrm{Ind}_N^G(\chi_i \otimes \eta)$ of degree p for $i = 1, \ldots, m$, and the representation $\psi := (\chi_0 \otimes \rho_1) \oplus \cdots \oplus (\chi_0 \otimes \rho_{p-1})$ of degree $p-1$.*

According to [1, Proposition 13.6.1] the rational irreducible representations of G correspond canonically and bijectively to a set of G-stable abelian subvarieties of the Jacobian JZ of Z such that the addition map is an isogeny. If the abelian subvariety of JZ corresponding to the rational irreducible representation ρ of G is denoted by J_ρ, the isotypical decomposition of JZ is the isogeny given by the addition map

$$J_{\rho_0} \times J_\psi \times J_{\theta_1} \times \cdots \times J_{\theta_m} \to JZ.$$

Furthermore, according to [1, Proposition 13.6.2] and [2], for each rational irreducible representation ρ of G there exist abelian subvarieties B_ρ of J_ρ such that $B_\rho^{n_\rho}$ is isogenous to J_ρ, with

$$n_\rho = \frac{\dim V_\rho}{m_\rho},$$

where V_ρ is a complex irreducible representation of G Galois associated to ρ and m_ρ is the Schur index of V_ρ.

The subvarieties B_ρ are, in general, determined only up to isogeny, with $B_{\rho_0} = J_{\rho_0} = J(Z/G)$. In our case, we have $m_{\rho_0} = m_\psi = m_{\theta_j} = 1$, $\dim(V_0) = \dim(V_\psi) = 1$ and $\dim(V_{\theta_j}) = p$, hence

$$B_\psi = J_\psi \quad , \quad J_{\theta_j} \sim B_{\theta_j}^p \quad , \quad J_{\rho_0} = J\mathbb{P}^1 = 0$$

where $\sim$ denotes isogeny.

Furthermore, it follows from [2, Corollary 5.6] that

$$B_{\theta_j} \sim P(Y_j/X) \;\text{ and }\; J_\psi \sim J(X).$$

Therefore the group algebra decomposition of JZ is given by

$$JX \times \prod_{j=1}^{m} P(Y_j/X)^p \to JZ.$$

3. The isogeny α

Let the notation be as in Section 1 and for $i = 1, \ldots, m$ denote

$$\nu_i : Z \to Y_i \quad \text{and} \quad \mu : Z \to T,$$

the maps of diagram (2.6), so that $\nu_i^* : JY_i \to JZ$ and $\operatorname{Nm}\mu : JZ \to JT$ are the induced homomorphisms of the corresponding Jacobians. Then the addition map gives a homomorphism

$$\alpha := \sum_{i=1}^{m} \operatorname{Nm}\mu \circ \nu_i^* : \prod_{i=1}^{m} P(Y_i/X) \to JT. \tag{3.1}$$

According to Corollary 2.9, $\prod_{i=1}^{m} P(Y_i/X)$ and JT are of the same dimension. The aim of this section is the proof of the following theorem.

Theorem 3.1. *$\alpha : \prod_{i=1}^{m} P(Y_i/X) \to JT$ is an isogeny with kernel contained in the 2^{p-2}-division points.*

For this we use the following result (for the proof see [8, Corollary 2.7]).

Proposition 3.2. *Let $f : Z \to X := Z/N$ be a Galois cover of smooth projective curves with Galois group N and $H \subset G$ a subgroup. Denote by $\nu : Z \to Y := Z/H$ and $\varphi : Y \to X$ the corresponding covers. If $\{g_1, \dots, g_r\}$ is a complete set of representatives of G/H, then we have*

$$\nu^*(P(Y/X)) = \{z \in JZ^H \mid \sum_{i=1}^{r} g_i(z) = 0\}^0.$$

Now denote for $i = 1, \dots, m$,

$$A_i := \nu_i^*(P(Y_i/X))$$

and let

$$A := \sum_{i=1}^{m} A_i \quad \text{and} \quad B := \mu^*(JT).$$

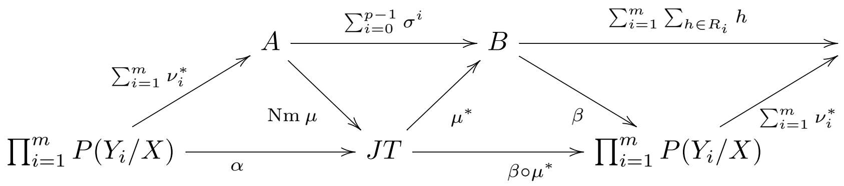
Recall from (2.4) that $G = N \rtimes P$ with

$$N = \left\{ \prod_{i=1}^{p-1} s_i^{j_i} \mid 0 \le j_i \le 1, i = 1, \dots, p-1 \right\} \quad \text{and} \quad P = \langle \sigma \rangle$$

with s_i and σ as in Section 1. The group P acts by conjugation on the elements of N by

$$\sigma^{-1} s_i \sigma = s_{i+1} \quad \text{for} \quad i = 1, \dots, p-1 \quad \text{with} \quad s_p = \prod_{i=1}^{p-1} s_i. \tag{3.2}$$

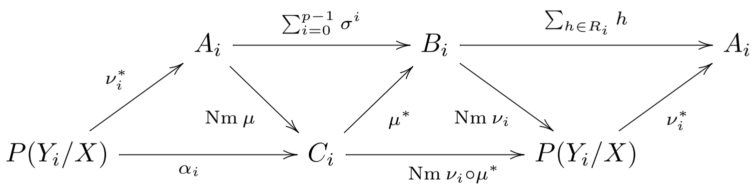
Recall furthermore that R_i is the subgroup of N giving the cover $Y_i \to X$. Then it is easy to see that we have the following commutative diagram



(3.3)

with $\beta = (\mathrm{Nm}\,\nu_1, \mathrm{Nm}\,\nu_2, \ldots, \mathrm{Nm}\,\nu_m)$.

For $i = 1, \ldots, m$ consider the following subdiagram



(3.4)

with $\alpha_i := \mathrm{Nm}\,\mu \circ \nu_i^*$, $C_i := \mathrm{Nm}\,\mu(A_i)$ and $B_i := \mu^*(C_i)$.

Proposition 3.3. *For $i = 1, \ldots, m$ the map $\mathrm{Nm}\,\nu_i \circ \mu^* \circ \alpha_i : P(Y_i/X) \to P(Y_i/X)$ is multiplication by 2^{p-2}.*

Proof. Since $\nu_i^* : P(Y_i/X) \to A_i$ is an isogeny, it suffices to show that the composition

$$\Phi_i := \sum_{h \in R_i} h \circ \sum_{i=0}^{p-1} \sigma^i : A_i \to A_i$$

is multiplication by 2^{p-2}.

Now from Proposition 3.2 we deduce

$$A_i = \{z \in JZ \mid hz = z \text{ for all } h \in R_i \text{ and } nz = -z \text{ for all } n \in N \setminus R_i\}^0 \tag{3.5}$$

since any $n \in N \setminus R_i$ induces the non-trivial involution of Y_i/X and A_i is the image of $P(Y_i/X)$.

Now for any $z \in A_i$,

$$\Phi_i(z) = \sum_{h \in R_i} h(z) + \sum_{h \in R_i} h \sum_{k=1}^{p-1} \sigma^k(z).$$

By equation (3.5) we have

$$\sum_{h \in R_i} h(z) = |R_i| z = 2^{p-2} z$$

and for $k = 1, \dots, p-1$,

$$\sum_{h \in R_i} h\sigma^k(z) = \sigma^k \sum_{h \in \sigma^{-k} R_i \sigma^k} h(z) = 0,$$

since $R_i \neq \sigma^{-k} R_i \sigma^k$ and considering that half of the elements of the subgroup $\sigma^k R_i \sigma^k$ belong to R_i, hence fix z, and the other half belongs to $N \setminus R_i$ and hence sends z to $-z$. Together this completes the proof of the proposition. □

Proof of Theorem 3.1. Since

$$\beta \circ \mu^* \circ \alpha = \prod_{i=1}^{m} (\mathrm{Nm}_{\nu_i} \circ \mu^* \circ \alpha_i),$$

Proposition 3.2 implies that $\beta \circ \mu^* \circ \alpha$ is multiplication by 2^{p-2}. In particular α has finite kernel. But according to Corollary 2.9, $\prod_{i=1}^{m} P(Y_i/X)$ and JT have the same dimension. So α is an isogeny. □

Corollary 3.4. *Given any positive integer N, there exist smooth projective curves Y whose Jacobian is isogenous to the product of $m \geq N$ principally polarized abelian varieties of the same dimension.*

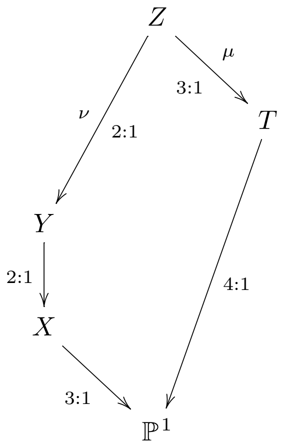
Proof. Choose a prime p such that $\frac{1}{p}(2^{p-1} - 1) \geq N$. This is equivalent to $2^{p-1} > pN$. Hence there are infinitely many primes with this property. According to Theorem 3.1, the Jacobian JT has the property of the corollary. □

We thank the referee for suggesting the following remark and Elham Izadi for the contents of it.

Remark 3.5. There is a slight relation of Corollary 3.4 and a question of Ekedahl and Serre [3], whether for any positive integer g there is a smooth curve of genus g whose Jacobian is isogenous to a product of elliptic curves. Izadi showed in [4] that, if there is a complete subvariety of codimension g in the moduli space, then there exist smooth curves of genus g whose Jacobian is isogenous to the product of elliptic curves. As was later proved by Keel and Sadun however in [5], there are no such subvarieties in characteristic 0.

4. The case $p = 3$

In this case we have $m = 1$, so let $Y_1 =: Y$, $\nu_1 =: \nu$ and $A_1 =: A$. Moreover, the subgroup N is the Klein group of order 4. Diagram (2.6) simplifies to



(4.1)

Theorem 4.1. *The map $\alpha = \nu^* \circ \operatorname{Nm}\mu : P(Y/X) \to JT$ is an isogeny with kernel the group $P(Y/X)[2]$ of all two-division points.*

Proof. From Theorem 3.1 we know that $\ker\alpha \subseteq P(Y/X)[2]$. On the other hand, μ^* is injective, since $\mu : Z \to T$ is ramified. Hence from diagram (3.4) we have $\ker(\operatorname{Nm}\mu|_A) = \ker(1 + \sigma + \sigma^2)|_A)$. So we get

$$\ker\alpha = \{z \in P(Y/X)[2] \mid (1 + \sigma + \sigma^2)(\nu^*(z)) = 0\}.$$

Let $\gamma : Y \to X$ denote the double covering and $\epsilon : Z \to X$ the composition

$$\epsilon = \gamma \circ \nu.$$

Since N is a normal subgroup of G, the automorphism σ descends to an automorphism $\overline{\sigma} : X \to X$, also of order 3. This is the automorphism giving the cyclic covering $X \to \mathbb{P}^1$.

Suppose η is the two-division point of JX giving the double cover γ and let $\eta^\perp$ be the subgroup of $JX[2]$ orthogonal with respect to the Weil form $e_{2\lambda}$ associated to twice the canonical polarization λ of JX. Then from [6] we know that

$$P(Y/X)[2] = \gamma^*(\eta^\perp).$$

This gives

$$\begin{aligned}\ker\alpha &= \gamma^*\{x \in \eta^\perp \mid (1 + \sigma + \sigma^2)\epsilon^*(x) = 0\} \\ &= \gamma^*\{x \in \eta^\perp \mid \epsilon^*(1 + \overline{\sigma} + \overline{\sigma}^2)(x) = 0\}.\end{aligned}$$

But $JX = \ker(1 + \overline{\sigma} + \overline{\sigma}^2)$. In particular for all $x \in \eta^\perp$ we have $\epsilon^*(1 + \overline{\sigma} + \overline{\sigma}^2)(x) = 0$. Together this implies $\ker\alpha = P(Y/X)[2]$. □

As an immediate consequence we get a version of the trigonal construction in the special case of an étale cover of a cyclic trigonal cover $X \to \mathbb{P}^1$.

Corollary 4.2. *Let the notation be as in Theorem 4.1. The isogeny $\alpha : P(Y/X) \to JT$ induces an isomorphism of principally polarized abelian varieties*

$$\overline{\alpha} : \widehat{P(Y/X)} \to JT$$

where $\widehat{\ }$ *denotes the dual abelian variety.*

Proof. Let λ_P denote the polarization on $P(Y/X)$ induced by the canonical polarization of JY. It is twice a principal polarization. According to Theorem 4.1, α has kernel $P(Y/X)[2]$ which coincides with the kernel of the polarization λ_P. Hence α factors as follows, with $\overline{\alpha}$ an isomorphism,

$$\begin{array}{ccc} P(Y/X) & \xrightarrow{\alpha} & JT \\ {\scriptstyle \lambda_P}\downarrow & \nearrow {\scriptstyle \overline{\alpha}}\ \simeq & \\ \widehat{P(Y/X)} & & \end{array} \tag{4.2}$$

It remains to show that $\overline{\alpha}$ respects the principal polarizations. If we denote by λ_1 the polarization of $\widehat{P(Y/X)}$ induced via $\overline{\alpha}$ from the canonical polarization λ_{JT} of JT, we may complete diagram 4.2 to the following one.

$$\begin{array}{ccc} P(Y/X) & \xrightarrow{\alpha} & JT \\ {\scriptstyle \lambda_P}\downarrow & \nearrow {\scriptstyle \overline{\alpha}}\ \simeq & \downarrow {\scriptstyle \lambda_{JT}} \\ \widehat{P(Y/X)} & & \widehat{JT} \\ {\scriptstyle \lambda_1}\downarrow & \swarrow {\scriptstyle \widehat{\overline{\alpha}}}\ \simeq & \\ P(Y/X) & & \end{array}$$

It now follows from the commutativity of this diagram that λ_1 is principal and that $\ker(\lambda_1 \circ \lambda_P) = P(Y/X)[2]$. Hence λ_1 is the canonical principal polarization on $\widehat{P(Y/X)}$ as claimed. □

5. Estimate of the kernel of α for odd p

We show that the same proof as in the last section gives for any odd prime p a lower bound for the order of the kernel of the isogeny $\alpha : \prod_{i=1}^{m} P(Y_i/X) \to JT$. We have the following result.

Proposition 5.1. *With the notation of above we have for any odd prime p,*

$$\prod_{i=1}^{m} P(Y_i/X)[2] \subset \operatorname{Ker}\alpha \subset \prod_{i=1}^{m} P(Y_i/X)[2^{p-2}].$$

Furthermore, for $p > 3$, $\operatorname{Ker}\alpha$ cannot be equal to $\prod_{i=1}^{m} P(Y_i/X)[2]$.

Proof. For the first assertion it suffices to see that $\operatorname{Ker}\alpha_i$ contains $P(Y_i/X)[2]$. But since μ^* is injective, μ being ramified of prime degree, it follows from diagram (3.4) and Theorem 3.1 that

$$\operatorname{Ker}\alpha_i = \{z \in P(Y_i/X)[2^{p-2}] \mid \sum_{i=0}^{p-1} \sigma^i(\nu_i^*(z)) = 0\}$$

Hence for the proof of the first assertion it suffices to show that for any $z \in P(Y_i/X)[2]$ we have

$$\sum_{i=0}^{p-1} \sigma^i(\nu_i^*(z)) = 0.$$

This follows with the same proof as in the proof of Theorem 4.1 for $p = 3$.

Finally, if we had $\operatorname{Ker}\alpha = \prod_{i=1}^{m} P(Y_i/X)[2]$, the same proof as for Corollary 4.2 would provide an isomorphism of principally polarized abelian varieties $\prod_{i=1}^{m} \widehat{P(Y_i/X)} \simeq JT$. For $p > 3$, i.e. $m > 1$, this contradicts the fact that the canonical polarization of JT is irreducible. □

6. The case $p = 2$

Let $Y \to X$ be an étale double covering of a double covering $X \to \mathbb{P}^1$. According to Corollary 2.2, the composition $Y \to \mathbb{P}^1$ is Galois, with Galois group the Klein group

$$G = \langle r, s \mid r^2 = s^2 = (rs)^2 = 1\rangle.$$

Denoting $Y_r := Y/\langle r\rangle$ and similarly Y_s and Y_{rs}, we have the following diagram of double coverings,

$$\begin{array}{ccccc} & & Y & & \\ & \swarrow^{\nu_s} & \downarrow^{\nu_r} & \searrow^{\nu_{rs}} & \\ X = Y_s & & Y_r & & Y_{rs} \\ & \searrow_{2:1} & \downarrow & \swarrow & \\ & & \mathbb{P}^1 & & \end{array} \tag{6.1}$$

We assume that ν_s is étale and $Y_s \to \mathbb{P}^1$ is ramified over 2β points of $\mathbb{P}_1$, with $\beta \geq 3$ (so that $\dim P(Y/Y_s) > 0$). Each branch point of $Y_s \to \mathbb{P}^1$ is a branch point of exactly one of the maps $Y_r \to \mathbb{P}^1$ and $Y_{rs} \to \mathbb{P}^1$. So if $2\beta_r$ respectively $2\beta_{rs}$ denote the number of branch points of $Y_r \to \mathbb{P}^1$ respectively $Y_{rs} \to \mathbb{P}^1$, we have

$$\beta = \beta_r + \beta_{rs}.$$

The genera of the curves are:

$$g(Y_s) = \beta - 1; \quad g(Y) = 2\beta - 3; \quad g(Y_r) = \beta_r - 1; \quad g(Y_{rs}) = \beta_{rs} - 1.$$

In particular, $\dim P(Y/Y_s) = g(Y_r) + g(Y_{rs})$.

Proposition 6.1. *The following map is an isogeny,*

$$\alpha : JY_r \times JY_{rs} \to P(Y/Y_s), \quad (x_1, x_2) \mapsto \nu_r^*(x_1) + \nu_{rs}^*(x_2)$$

with kernel consisting at most of two-division points.

Proof. First we claim that $\mathrm{Im}(\alpha) \subset P(Y/Y_s)$. Note first that the automorphism s descends to an automorphism $\overline{s}$ of Y_r and we have for any $x \in JY_r$

$$s(\nu_r^*(x)) = \nu_r^*(\overline{s}(x)) = -\nu_r^*(x)$$

where the last equation follows from Proposition 3.2. An analogous equation is valid for ν_{rs}^*. So we have

$$(1+s)(\alpha(x_1, x_2)) = (1+s)(\nu_r^*(x_1) + \nu_{rs}^*(x_2)) = x_1 - x_1 + x_2 - x_2 = 0,$$

which implies the assertion.

It remains to show that $\ker \alpha$ consists of 2-division points, since $g(Y_r) + g(Y_{rs}) = \dim P(Y/Y_s)$. For this it suffices to show that the composed map

$$JY_r \times JY_{rs} \xrightarrow{\nu_r^* + \nu_{rs}^*} JY \xrightarrow{(\mathrm{Nm}\,\nu_r, \mathrm{Nm}\,\nu_{rs})} JY_r \times JY_{rs}$$

is multiplication by 2. But $\mathrm{Nm}\,\nu_r \circ \nu_r^* = \deg \nu_r = 2$ and the same is valid for ν_{rs}. This completes the proof of the proposition. □

Proposition 6.1 implies

$$\begin{aligned}\ker \alpha &= \{(x_1, x_2) \in JY_r[2] \times JY_{rs}[2] \mid \nu_r^*(x_1) = \nu_{rs}^*(x_2)\} \\ &= (\nu_r^* \times \nu_{rs}^*)^{-1}\{(x, x) \in JY \times JY \mid x \in \nu_r^* JY_r[2] \cap \nu_{rs}^* JY_{rs}[2]\}\end{aligned}$$

Since ν_r and ν_{rs} are ramified, the homomorphisms ν_r^* and ν_{rs}^* are injective. Hence we get

$$\deg \alpha = |\nu_r^* JY_r[2] \cap \nu_{rs}^* JY_{rs}[2]|. \tag{6.2}$$

The following theorem is due to Mumford (see [6, p. 356]).

Theorem 6.2. *Let the notation be as above. Then we have:*

(i) *the map*

$$\alpha : JY_r \times JY_{rs} \to P(Y/Y_s)$$

is an isomorphism;

(ii) *the isomorphism α respects the canonical principal polarizations.*

Proof. (i): According to (6.2) it suffices to show that the images of $JY_r[2]$ via ν_r^* and $JY_{rs}[2]$ via ν_{rs}^* in $JY[2]$ intersect only in $0 \in JY$. Now, fixing a theta characteristic of JY, the 2-division points of JY correspond in a natural way bijectively to the theta characteristics of Y. An analogous statement is valid for JY_r and JY_{rs}. Using this, the assertion follows from the fact that the theta characteristics of Y which are pullbacks from theta characteristics of Y_r are disjoint from those which are pullbacks from theta characteristics of Y_{rs}.

But this follows from the fact that, according to what we have said right after the diagram 6.1, the branch points $b_1, \dots, b_{2\beta}$ of $Y_s \to \mathbb{P}^1$ can be enumerated in such a way that $b_1, \dots, b_{2\beta_r}$ are the branch points of $Y_r \to \mathbb{P}^1$ and that $b_{2\beta_r+1}, \dots b_{2\beta}$ are the branch points of $Y_{rs} \to \mathbb{P}^1$. For this, note only that all theta characteristics of a hyperelliptic curve are sums of ramification points of the hyperelliptic covering (see for example [7, Section III, 5]).

(ii): From the proof of Proposition 6.1 we know that the composition

$$JY_r \times JY_{rs} \xrightarrow{\alpha} P(Y/Y_s) \xrightarrow{\gamma} JY_r \times JY_{rs}$$

with $\gamma := (\mathrm{Nm}\,\nu_r, \mathrm{Nm}\,\nu_{rs})$, is multiplication by 2. If $\theta := \theta_{JY \times JY_{rs}}$ denotes the canonical polarization of $JY_r \times JY_{rs}$, this implies that $(\gamma \circ \alpha)^{-1}(\theta) = 4\theta$ (see [1, Corollary 2.3.6]).

Since α is an isomorphism, it follows that $\gamma^{-1}(\theta)$ is the fourth power of a principal polarization, say $\gamma^{-1}(\theta) = 4\Xi$.

Now $\alpha : JY_r \times JY_{rs} \to P(Y/Y_s)$ is an isomorphism. The canonical principal polarization of JY restricts to $\nu_r^*(JY_r)$ as twice a principal one, and to $\nu_{rs}^*(JY_{rs})$ as twice a principal one, the restriction to $P(Y/Y_s)$ is 2Ξ. Then (ii) follows from the fact that the map α is G-equivariant, since both varieties are the eigen-subvarieties of -1 for the same element of G, namely σ. □

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