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RESEARCH ARTICLE

Joint Lagrangian Relaxation and Dynamic Programming Approach for a Multi-Period Multi-Product Location-Inventory Problem

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ABSTRACT We propose a multi-period multi-product location-inventory model considering an (R, s, S) periodic review policy and modular stochastic capacity constraints. The objective is to define which regional warehouses to open, maintain, operate, or close, which products to assign, and which customers should be supplied by the chosen distribution centers. Additionally, it involves setting order sizes and reorder points to minimize system costs while meeting service level conditions. This problem is structured as a non-convex mixed integer nonlinear programming model. We propose a Lagrangian relaxation algorithm and the subgradient approach as a solution method. We relax the customer allocation, distribution center variance, and demand constraints. Then, the relaxed problem is decomposed into location subproblems for each warehouse, in which inventory subproblems are embedded. So, each location subproblem is discomposed for each period and exactly solved through dynamic programming. Computational experiments prove that the presented approach gives solutions close to optimal and low gaps in a low computational time. Additionally, it exhibits meaningful incidences in the decisions on inventory control policy, facility location decisions, total costs, and risk pooling effects for different review intervals.

INDEX TERMS Dynamic programming, Lagrangian relaxation, location-inventory problem.

I. INTRODUCTION

Today's supply chains face competitive markets characterized by a high uncertainty in demand for multiple products in the short and long term. This situation has generated the need for a flexible, reliable, and fast response supply chain network design (SCND), adjusting over time to satisfy a changing demand while maintaining low logistics costs. Thus, the location, capacity, and number of facilities (e.g., regional warehouses) of a SCND could adjust as needed to the variability of the product demand of each geographical area. Thus, relevant issues include opening new warehouses in zones with increasing demand growth and closing distribution centers in areas with declining demand. Also, capacity modifications such as removing or adding capacity in the opened facilities is another essential strategy to appropriately fit demand variations [1].

Additionally, strategic decisions, such as facility location and capacity planning, must integrate simultaneously with tactical decisions, such as inventory control policies, to optimize the supply chain total operation cost while meeting demands at the service level required by customers. Accordingly, an integrated multi-period or "dynamic" SCND is needed. This dynamic supply chain network design (DSCND) concept considers the facility opening or closing decision in each period, depending on whether that facility was previously operated.

Addressing the real-world relevance of our research, a few researchers have analyzed the location-inventory problem (LIP) in a multi-period scenario, i.e., on the joint formulation

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of facility location and inventory control decisions and how they change over strategic and tactical periods. Most of these authors have studied dynamic LIP considering a deterministic demand (e.g., [2]). However, real-world supply chains inherently face an uncertain demand, the variance of which is directly equivalent to the safety stock necessary to achieve a predefined service level. This uncertainty, when integrated with location decisions, configures the risk pooling effect, which impacts network configuration and, therefore, its performance. This effect cannot be captured in a study that only considers deterministic demand. To address this issue, some authors have studied the dynamic LIP with stochastic demand but in a single-commodity scenario (e.g., [3]). However, most of today's logistics networks must address a multi-product scenario or multiple categories of products, such as miscellaneous products, non-perishable foods, perishable items, and other specific categories (e.g., Walmart). Few researchers have tackled the dynamic multi-product LIP with stochastic demand [4], [5], [6].

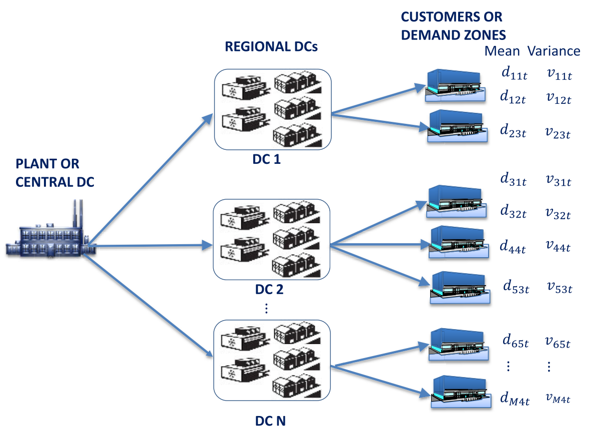

FIGURE 1. Multi-period multi-product distribution network.

The problems discussed so far consider an aggregated warehouse capacity, which must be given uniquely to all commodities or categories. However, such aggregation is no further realistic if various products impose distinct capacity requirements [7]. This is the case when some products must be stored under certain temperature conditions to preserve their natural properties, e.g., an enterprise resource planning software that defines storage capacity at the "category level" such as frozen, refrigerated, dry, and others [8]. Thus, we must consider a dynamic multi-product LIP model, for example, where the capacities of warehouses, demand, and flows are separated concerning some homogeneous product groups or categories. The primary motivation of this study is to address the gaps that have been previously observed in the literature.

This paper introduces a dynamic multi-product location-inventory problem (DMLIP). We address a three-stage logistic network, wherein a central warehouse supplies several regional DCs, which then provide a retailer's collection, each characterized by stochastic demand in multi-period and multi-product scenarios, as is shown in Fig. 1. We consider that each demand zone must be served by only one DC for each product, following a single source policy. We are adopting an (*R, s, S*) policy for each warehouse. The model formulation revolves around a modular decision regarding the opening and closing inventory capacity for each product category at the distribution centers (DCs) rather than considering an aggregate capacity for all products. We develop a location-inventory model that incorporates expected inventory, ordering costs, stochastic inventory capacity constraints, all under a periodic review control policy. We model this location-inventory model by studying the safety stock, ordering quantity, working inventory, and highest inventory levels for every possible distribution center. This mixed integer nonlinear programming (MINLP) model is classified as NP-Hard because it is an extension of the multi-commodity location-inventory model, which already is NP-hard [9].

We introduce Lagrangian relaxation and the subgradient procedure as a solution strategy to address the high intractability of the model studied. We relax three sets of constraints of the primal problem as a part of the decomposition approach. Then, we break down the relaxed problem into separate subproblems for each distribution center and product. Later, each subproblem is further discomposed into location and inventory subproblems. Then, we solve the relaxed subproblems by dynamic programming approach. The Lagrangian heuristic comprises three procedures: the selection of distribution centers, a customers' greedy assignment, and local search improvements. Finally, this model is solved using two main instances: the first dataset of 20 potential warehouse location sites, 20 customers, and 5 products, and the second dataset of 20 potential warehouse location sites, 40 customers, and 5 products. Our proposed Lagrangian Relaxation algorithm obtains low CPU times with reduced duality gaps, thus close to optimal solutions.

Based on the information provided, we have summarized the main contributions of this paper that distinguish it from other relevant studies as follows:

- It introduces a multi-period multi-product location-inventory problem with modular stochastic capacity constraints under periodic review policy.
- It develops an MINLP model for solving the problem.
- It solves the relaxed subproblems exact solutions are obtained using dynamic programming.
- It shows that the proposed algorithm obtains close to optimal solutions in a short time.

This research is organized as follows: Section II reviews the relevant dynamic location-inventory models found in the literature. Section III presents the formulation of the dynamic location-inventory model with modular stochastic capacity constraints under periodic review control policy. Section IV outlines the proposed Lagrangian relaxation and dynamic programming solution approach. Section V introduces and examines computational results. In Section VI, we conclude by incorporating insights from a managerial perspective and providing recommendations for future work.

II. LITERATURE REVIEW

Various researchers have proposed diverse models for the past thirty years to approach the location-inventory problem (LIP). These models have become increasingly complex in capturing the uncertainty and time evolution of today's SCND needs. This evolution continues since there are still many gaps in the current industrial requirements. We can classify these LIPs as follows: static single commodity LIP (SSLIP), static multi-product LIP (SMLIP), dynamic single commodity LIP (DSLIP), and dynamic multi-product LIP (DMLIP). Some reviews on previous LIP models exist, e.g., [10], [11], and [12]. There is a vast body of literature on joint location-inventory topics; therefore, we only consider relevant research that includes DSLIP and DMLIP models. A list of these articles, including their main model's structure, objective function cost components, and solution approach, is presented in Table 1.

Reference [10] propose a DMLIP model formulated as mixed integer linear programming (MILP) with deterministic demand. This MILP model includes facility location and relocation of plants and DCs, considering the investment capital budget and deterministic inventory capacity constraints. Note that these inventory capacities are aggregated for the entire facility, not each product. They minimize the objective function that includes production, transportation, inventory maintenance, and fixed operating costs for facilities. They test different formulations of the problem and solve them using a branch-and-bound algorithm of commercial software.

Reference [13] study a DSLIP with deterministic demand, allowing warehouse inventory storage between consecutive periods. In addition, the facility location problem considers opening and closing warehouses at two echelons during the planning period. The problem is structured as a MILP model, aiming to optimize the total costs, which include inventory holding and transportation costs for commodities aside from operating and fixed costs associated with warehouses. Also, they propose a Lagrangian relaxation algorithm to solve this problem using a decomposition scheme: they first break down the problem by tiers, then they disaggregate the obtained subproblems into one problem for each warehouse. Finally, they further decompose these problems into one problem for each period.

Reference [14] address the planning and design of a production and distribution problem, which includes the opening, closing, or extension of plants and warehouses, supplier selection, and bill of materials. They consider a network with four leading echelons constituted by plants, suppliers, warehouses, and retailers. They propose a MILP model with deterministic customer demand, differentiating between a public distribution center, rented by the company, and a private distribution center, owned by the company. The public warehouse status could change multiple times during the planning period. They solve the model using a commercial solver.

Reference [15] study a DSLIP with stochastic demand, integrating a production-distribution location-allocation system and incorporating service level, production rates, and safety stock optimization. They propose a MINLP model and then linearize and solve it using a recursive procedure, which determines a lower bound of safety stock levels, to achieve a feasible solution to the model.

Reference [16] present a multiple-echelon DMLIP, considering several CPU times for strategic and tactical decisions. The logistic network's capacity increase in the MILP model is planned considering accumulated net profits and funds provided by external sources. Also, they consider some characteristics, such as public distribution centers, the minimum and maximum values of facilities utilization indicators, and possible sites for establishing private distribution centers. Finally, they solve the model by applying a Lagrangian relaxation solution approach.

Reference [4] study a DMLIP formulated as MINLP and MILP models for supply chain network design of the steel industry. They consider a supply chain with uncertain demand and three leading echelons: iron ore mines serving as suppliers, raw steel manufacturers as producers, and retail steel business as customers. The main decisions addressed are facility location, production planning, and inventory holding. In addition, the authors contemplated a possible production capacity along with shared safety and emergency stocks to achieve the required service level. They solve these models using a commercial solver.

Reference [5] address a stochastic closed-loop DMLIP under facility disruption risks, including selecting warehouses and reworking centers, allocating centers, and making inventory control decisions. To solve this problem, they propose Multi-Objective Particle Swarm Optimization (MOPSO) and a hybrid meta-heuristic approach founded on Non-dominated Sorting Genetic Algorithm (NSGA-II).

Reference [17] address a multi-stage DMLIP with deterministic demand. The logistic network considers manufacturing plants, distribution centers, and demand zones. Strategic decisions consider location and capacity extension of facilities subject to a budget constraint. The capacitated distribution centers can be opened only once during the planning period. First, the production plant location is determined; then, each open manufacturing plant's capacity is estimated for each production process. The level of production, commodity flows, inventory levels of finished products at warehouses, and products' prices correspond to tactical decisions. The authors propose an MILP model that includes a set of strategic and tactical periods whose objective function is net income maximization. They solve it using a simulated annealing (SA) algorithm and a heuristic established in linear relaxation.

Reference [6] tackle a three-tier pharmaceutical logistic network design problem, in which the supply chain is constituted of various manufacturers, warehouses, and demand

TABLE 1. Dynamic location-inventory models present in the literature.

Paper	Model								Objective function cost components								Solution approach (g)
	Model Type (a)	Location (b)	Inventory Policy (c)	Demand type (d)	Inventory capacity constraint (d)	Ordering capacity constraint (d)	Commodity (e)	Periods (f)	Fixed locating cost	Fixed closing cost	Fixed operating cost	Fixed operating cost for commodity	Ordering cost	Inventory holding cost	Safety stock cost	Transp. cost	
[10]	MILP	P, DC		D	D		M	S	✓		✓			✓		✓	Commercial solver
[13]	MILP	P, DC		D	D		M	S	✓	✓	✓			✓		✓	Lagrangian relaxation
[14]	MILP	P, DC		D	D		M	S	✓	✓	✓			✓		✓	Commercial solver
[15]	MINLP	DC	C	S	D		S	S			✓		✓	✓	✓	✓	Recursive procedure
[16]	MILP	P, DC		D	D		M	S, T	✓		✓	✓		✓		✓	Lagrangian relaxation
[4]	MILP, MINLP	P, DC	C	S	D		M, B	S	✓			✓		✓		✓	Commercial solver
[5]	ILP	DC	C	S			M	S	✓				✓	✓	✓	✓	MOPSO; NSGA-II
[17]	MILP	P, DC		D	D		M	S, T	✓	✓	✓	✓		✓		✓	Simulated annealing
[6]	MINLP	P, DC	C	S	S		M	S	✓				✓	✓		✓	Commercial solver
[3]	SP	DC	P	S			S	S, T	✓				✓	✓	✓	✓	Sample average approximation
[2]	ILP	DC		D			M	S	✓					✓		✓	Matheuristic
This paper	MINLP	DC	P	S	S	S	M	S	✓	✓	✓	✓	✓	✓	✓	✓	Lagrangian relaxation/DP

(a) MILP: Mixed-Integer Linear Programming; MINLP: Mixed-Integer Non-Linear Programming; ILP: Integer Linear Programming; SP: Stochastic Programming
(b) P: Plant; DC: Distribution center.
(c) C: Continuous review policy; P: Periodic review policy.
(d) D: Deterministic; S: Stochastic.
(e) S: Single commodity; M: Multi-product; B: Bulk material.
(f) S: Strategic; T: Tactical.
(g) MOPSO: Multi-Objective Particle Swarm Optimization; NSGA-II: Non-dominated Sorting Genetic Algorithm; DP: Dynamic Programming.

zones. This DMLIP incorporates opening warehouses and plants, managing material flows into the network, and determining the optimal inventory policy, considering product perishability, all while minimizing the total network cost. The authors propose a model formulated as MILP considering an (s, Q) continuous review inventory control policy. The imprecise parameters are addressed through a possibilistic programming approach. They solve the model using a commercial solver based on real data from a case study and provide managerial insights by developing some sensitivity analyses.

Reference [3] study a DSLIP with uncertain demand in a multi-tier logistic network with multiple sources. They propose a two-stage stochastic programming model that optimizes the logistic network's total expected profit, incorporating an (R, s, S) policy with $R = 1$. Because this model is non-linear, they propose a linear approximation and utilize a sample average approximation approach to generate near-optimal solutions. Computational runs are developed, and the results demonstrate the efficiency of the linear approach of the (R, s, S) policy at the strategic level, giving resilient design solutions under stochastic scenarios.

Reference [2] investigate a joint location, manufacturing, distribution, and inventory problem. This DMLIP aims at satisfying deterministic demand within a delivery time window while minimizing the total supply chain costs. They compare the sequential versus the integrated optimization solution approach to solve the problem. They address the problem sequentially, using several commonly employed procedures that solve each part separately. The joint problem is tackled using both an exact procedure and a matheuristic method. Their computational tests compare the solution costs obtained from these two approaches, distinguish the advantages of an integrated procedure, and offer valuable managerial perspectives into the integration gains.

As the literature reviewed above shows, the DMLIP with modular stochastic capacity constraints under periodic review inventory control policy has not been addressed so far. Specifically, no study has incorporated stochastic ordering and inventory capacity constraints simultaneously into the model. So, the risk-pooling effect could not be analyzed in a multi-product and multi-period scenario until now, namely, how larger groupings of customers assigned to a DC and thus higher allocation variance demand decrease the safety stock level and, therefore, the total inventory costs. Nevertheless, relevant cost savings can be obtained by integrating these decisions simultaneously into the planning horizon. In this study, we present a multi-product, multi-period location-inventory model incorporating stochastic capacity constraints. This model includes a fixed cost component at each distribution center linked to a modular capacity of the products. Furthermore, we analyze this model utilizing a periodic review inventory policy. This allows us to assess the trade-off between the ordering and fixed costs and the risk pooling effect at a particular distribution center as more clients and products are allocated to that warehouse. Additionally, our model accounts for the reciprocal dependence between ordering, inventory levels, and inventory capacity

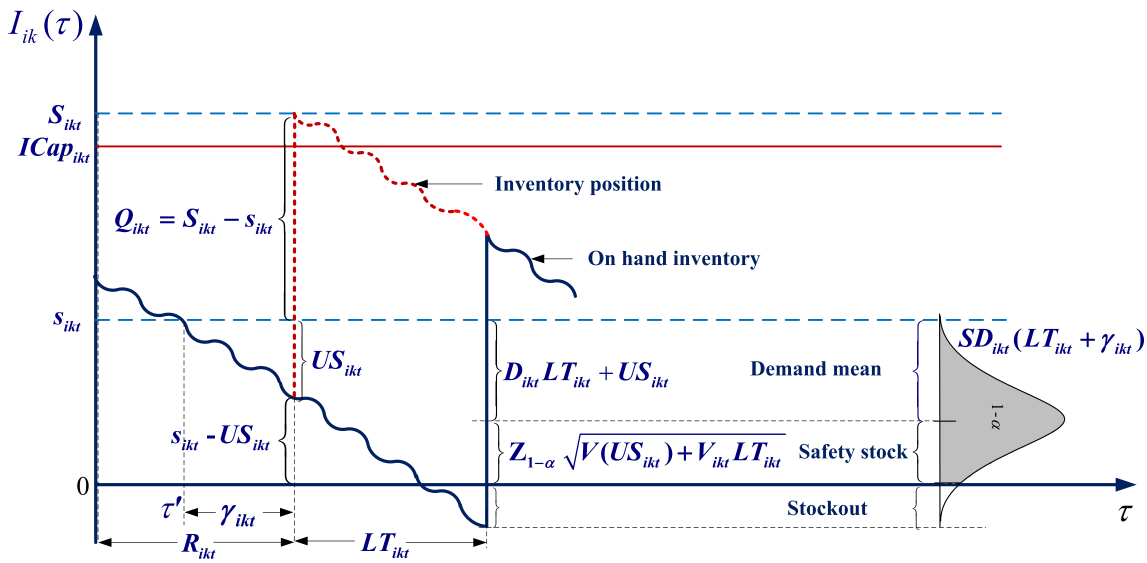

FIGURE 2. (R, s, S) inventory control policy considering periodic review.

at the distribution centers. Our contribution aims to explore these pertinent extensions of the dynamic multi-product location inventory problem (DMLIP) model. In the following section, we introduce the model formulation.

III. MODEL FORMULATION AND INVENTORY CONTROL

This section addresses capacity constraints, and an (*R, s, S*) inventory policy is applied to the dynamic multi-product facility location with a stochastic demand modeling approach. First, we focus on this control policy due to its prominence in industrial applications and literature. Furthermore, this control policy operates independently for each product. In addition, we assume that the inventory is not significant along the planning horizon at the end of each period since we consider only the long-term annual period relevant. Then, we introduce the formulation of the dynamic multi-product location-inventory model regarding an (*R, s, S*) control policy.

A. AN (R, S, S) INVENTORY CONTROL POLICY S

Unlike a continuous review policy, within an $(R_{ikt}, s_{ikt}, S_{ikt})$ control policy, the inventory on hand is reviewed each R_{ikt} period for each commodity k, time t,and warehouse i. Because of this, the exact moment when the inventory level effectively crosses the reorder point is not observable. Therefore, the order size for each commodity k, time t and warehouse i, must incorporate the undershoot magnitude (US_{ikt}). This denotes the quantity of items needed to be ordered additionally to $S_{ikt} - s_{ikt}$ achieve S_{ikt} quantity of inventory, as illustrated in Fig. 2. Thus, if the inventory level falls under the reorder point s_{ikt}, then an order is initiated to replenish the quantity to meet the desired level S_{ikt} (order-up-to-level). This article considers a periodic review inventory control policy in a multiple products and periods scenario. This inventory control policy structure incorporates the undershoot variance and average undershoot into safety stock and average demand expressions.

Considering a specific review period R_{ikt}, demand mean D_{ikt}, and variance V_{ikt} of a distribution center i for each product k and time t, the undershoot mean is estimated as follows:

$$US_{ikt}(D_{ikt}, V_{ikt}) = \frac{V_{ikt}}{2D_{ikt}} + \frac{D_{ikt}R_{ikt}}{2} \tag{1}$$

In addition, we define the variance of undershoots:

$$V\,(US_{ikt}) = \frac{D_{ikt}^2 R_{ikt}^2}{12} + \frac{V_{ikt}R_{ikt}}{2} - \frac{V_{ikt}^2}{4D_{ikt}^2} \tag{2}$$

It assumes that the inventory capacity constraint should be revised for each peak inventory moment (i.e., for each order period) with a known and constant probability $1 - \beta$ within the context of a periodic review policy, as follows:

$$\Pr(S_{ikt} - SD_{ikt}(LT_{ikt}) \leq ICap_{ikt}) \geq 1 - \beta \tag{3}$$

This constraint is redefined as a deterministic nonlinear constraint to ensure the satisfaction of the stochastic constraint:

$$\begin{aligned} Q_{ikt} + US_{ikt} + Z_{1-\alpha}\sqrt{V(US_{ikt}) + V_{ikt}LT_{ikt}} \\ + Z_{1-\beta}\sqrt{V_{ikt}LT_{ikt}} \leq ICap_{ikt} \end{aligned} \tag{4}$$

Additionally, we present a novel set of stochastic constraint for ordering that assures average ordering units, $\widehat{Q}_{ikt} = Q_{ikt} + US_{ikt}$ is at ordering capacity $QCap_{ikt}$ with a minimum value corresponding to the cumulative probability $1-\gamma$ as follows:

$$\Pr(\widehat{Q}_{ikt} \leq QCap_{ikt}) \geq 1 - \gamma \tag{5}$$

Also, this constraint is transformed into a deterministic nonlinear constraint:

$$Q_{ikt} + US_{ikt} + Z_{1-\gamma}\sqrt{V(US_{ikt})} \leq QCap_{ikt} \tag{6}$$

TABLE 2. Model decision variables.

Variable	Definition
X_{it}	It is equal to 1, if a DC is operating on site i during time period t, and 0 otherwise.
X'_{it}	It is equal to 1, if a DC is closed on site i at time period t, and 0 otherwise.
X''_{it}	It is equal to 1, if a DC is located at candidate site i at time period t, and 0 otherwise.
W_{ikt}	It is equal to 1, if a DC is operating on site i and is allocated product k during time period t, and 0 otherwise.
Y_{ijkt}	It is equal to 1, if warehouse i serves customer j for product k during time period t, and 0 otherwise.
Q_{ikt}	Order quantity at the DC i for product k during time period t (units).
D_{ikt}	Fulfilled demand by DC i for product k during time period t (units).
V_{ikt}	Variance in the demand met by DC i for product k during time period t (units).
US_{ikt}	Undershoot mean for DC i for product k during time period t (units).
$V(US_{ikt})$	Undershoot variance for DC i for product k during time period t (units).

TABLE 3. Parameters.

Parameter	Definition
N	Potential sites number to locate warehouses.
M	Customers number to be satisfied.
K	Products number.
T	Time periods number.
RC_{ikt}	Transportation unit cost to ship product k between the plant and the warehouse i during time period t (\$/unit).
TC_{ijt}	Fixed transportation cost between the warehouse i and the customer j during time period t (\$).
F_{it}	Fixed operating cost at candidate site i during time period t.
F'_{it}	Fixed closing cost at candidate site i at time period t.
F''_{it}	Fixed locating cost at candidate site i at time period t.
G_{ikt}	Fixed operating cost at site i for product k during time period t.
HC_{ikt}	Holding cost at site i for product k during time period t per time unit (\$/day).
OC_{ikt}	Fixed ordering cost at site i for product k during time period t per order (\$/order)
LT_{ikt}	Lead time when ordering from warehouse i for product k during time period t (days).
R_{ikt}	Review period of the warehouse i for product k during time period t (days).
d_{jkt}	Mean of the daily demand of product k at customer j during time period t.
v_{jkt}	Variance of the daily demand of product k at customer j during time period t.
$Z_{1\text{-}\alpha}$	Standard normal distribution value corresponding to the cumulative probability of 1-α.
$Z_{1\text{-}\beta}$	Standard normal distribution value corresponding to the cumulative probability of 1-β.
$Z_{1\text{-}\gamma}$	Standard normal distribution value corresponding to the cumulative probability of 1-γ.
$QCap_{ikt}$	Ordering capacity of product k at warehouse i during time period t.
$ICap_{ikt}$	Inventory capacity of product k at warehouse i during time period t.

Moreover, we assess ordering costs and expected inventory associated to order quantity, like the Economic Order Quantity (EOQ) model, focusing on minimizing the order quantity Q_{ikt} and the average undershoot US_{ikt}, as follows:

$$\frac{OC_{ikt}D_{ikt}}{(Q_{ikt}+US_{ikt}(D_{ikt},V_{ikt}))}+\frac{HC_{ikt}(Q_{ikt}+US_{ikt}(D_{ikt},V_{ikt}))}{2} \tag{7}$$

B. DYNAMIC MULTI-PRODUCT LOCATION-INVENTORY MODEL WITH AN (R, S, S) CONTROL POLICY

This section presents the Dynamic Multi-product Location-Inventory Model with Stochastic Constraints under Periodic Review (DMLIPR) as a Stochastic Non-Linear Mixed Integer Programming model. In the previous model, we address the challenge of inventory management and supply for multiple products across multiple periods, aiming to minimize total costs. We formulate this model as a Dynamic Multi-product Location-Inventory Problem, which simultaneously combines inventory control and location decisions. To formulate the multi-product facility location model structure, we were inspired by two kinds of Multi-product Location Problem (MLP). The first MLP is called the "multi-type model," which has been addressed by authors such as [18]. Also, we consider the MLP, "multi-activity model," studied by researchers such as [19] and [20]. The mathematical formulation encompasses the decision variables and parameters presented in Tables 2 and 3.

Using the above notations, the proposed DMILPR model is formulated as follows:

$$Min \sum_{i=1}^{N}\sum_{t=1}^{T}\left(F_{it}X_{it}+F'_{it}X'_{it}+\ F''{}_{it}X''{}_{it}\right)$$
$$+\sum_{i=1}^{N}\sum_{k=1}^{K}\sum_{t=1}^{T}G_{ikt}W_{ikt}+\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{k=1}^{K}\sum_{t=1}^{T}$$

$$\times\left(RC_{ikt}d_{jkt}+TC_{ijt}\right)Y_{ijkt}+\sum_{i=1}^{N}\sum_{k=1}^{K}\sum_{t=1}^{T}$$
$$\times\left(OC_{ikt}\frac{D_{ikt}}{\left(Q_{ikt}+US_{ikt}\right)}+HC_{ikt}\frac{\left(Q_{ikt}+US_{ikt}\right)}{2}\right)$$
$$+\sum_{i=1}^{N}\sum_{k=1}^{K}\sum_{t=1}^{T}HC_{ikt}(Z_{1-\alpha}\sqrt{V(US_{ikt})+V_{ikt}LT_{ikt}}) \tag{8}$$

$$\text{s. t. : } \sum_{i=1}^{N}Y_{ijkt}=1\ \forall j=1,\ldots,M,\ \forall k=1,\ldots,K,\ \forall t=1,\ldots,T \tag{9}$$

$$Y_{ijkt}\leq W_{ikt}\ \ \forall i=1,\ldots,N,\forall j=1,\ldots,M,\ \forall k=1,\ldots,K,\forall t=1,\ldots,T \tag{10}$$

$$W_{ikt}\leq X_{it}\ \ \forall i=1,\ldots,N,\forall k=1,\ldots,K,\forall t=1,\ldots,T \tag{11}$$

$$X_{it}-X_{it-1}=X''_{it}-X'_{it}\forall i=1,\ldots,N,\forall t=1,\ldots,T \tag{12}$$

$$Q_{ikt}+US_{ikt}+Z_{1-\alpha}\sqrt{V(US_{ikt})+V_{ikt}LT_{ikt}}+Z_{1-\beta}\sqrt{V_{ikt}LT_{ikt}}\leq ICap_{ikt}W_{ikt}\ \ \forall i=1,\ldots,N,\ \forall k=1,\ldots,K,\forall t=1,\ldots,T \tag{13}$$

$$Q_{ikt}+US_{ikt}+Z_{1-\gamma}\sqrt{V(US_{ikt})}\leq QCap_{ikt}\forall i=1,\ldots,N,\ \forall k=1,\ldots,K,\forall t=1,\ldots,T \tag{14}$$

$$D_{ikt}=\sum_{j=1}^{M}d_{jkt}Y_{ijkt}\forall i=1,\ldots,N,\forall k=1,\ldots,K,\ \forall t=1,\ldots,T \tag{15}$$

$$V_{ikt}=\sum_{j=1}^{M}v_{jkt}Y_{ijkt}\ \forall i=1,\ldots,N,\forall k=1,\ldots,K,\ \forall t=1,\ldots,T \tag{16}$$

$$US_{ikt}=\frac{V_{ikt}}{2D_{ikt}}+\frac{D_{ikt}R_{ikt}}{2}\ \forall i=1,\ldots,N,\forall k=1,\ldots,K,\ \forall t=1,\ldots,T \tag{17}$$

$$V\left(US_{ikt}\right)=\frac{D_{ikt}^{2}R_{ikt}^{2}}{12}+\frac{V_{ikt}R_{ikt}}{2}-\frac{V_{ikt}^{2}}{4D_{ikt}^{2}}\quad\forall i=1,\ldots,N,\ \forall k=1,\ldots,K,\forall t=1,\ldots,T \tag{18}$$

$$X_{it},X'_{it},X''_{it},W_{ikt},Y_{ijkt}\in\{0,1\}\qquad\forall i=1,\ldots,N,\ \forall j=1,\ldots,M,\forall k=1,\ldots,K,\forall t=1,\ldots,T \tag{19}$$

$$Q_{ikt},\ D_{ikt},V_{ikt},US_{ikt},V\left(US_{ikt}\right)\geq 0\forall i=1,\ldots,N,\ \forall k=1,\ldots,K,\forall t=1,\ldots,T \tag{20}$$

The total cost is minimized by the objective function (8). The first component corresponds to the operating fixed costs of DC plus fixed costs when closing DC and fixed costs of locating DCs. The second term is the fixed cost when opening DCs and allocating them to specific products. The third term accounts for the transportation costs between each DC and its assigned customers and the transportation and ordering costs between the plant and DCs for various products. The fourth term includes fixed, and inventory costs associated with the order size of products at DC. The fifth term corresponds to the safety stock cost at each DC. The constraints (9) state that only one warehouse supplies each customer for each product for each period. The constraints (10) assure each product k associated with each customer j for each period t can only be assigned to the open DC i allocated to the product. Meanwhile, constraints (11) guarantee that product k cannot be allocated to a DC at potential location i at period t as long as it is located first.

Inspired by [21], we included the constraints (12), which in conjunction with the first term of (8) ensures that appropriate operating, closing and opening fixed cost are charged. These constraints (12) assure the logic between opening variables (X''_{it}) and closing variables (X'_{it}) in function of operating variables from time period t (X_{it}) and previous time period $t-1$ (X_{it-1}). The minimization of costs obligates the binary values of X'_{it} and X''_{it} for each potential combination of values for X_{it} and X_{it-1} in constraints (12), as is shown in Table 4.

TABLE 4. Possible combinations of values for decision variables.

Combination	X_{it}	X_{it-1}	X'_{it}	X''_{it}
A	0	0	0	0
B	1	0	0	1
C	0	1	1	0
D	1	1	0	0

Note in (12) and combination A in Table 4 that if $X_{it}=0$ and $X_{it-1}=0$, i.e., if in location i in time period t and $t-1$ there does not exist an opened facility. Therefore, a facility cannot be opened ($X''_{it}=0$) or closed ($X'_{it}=0$) in time period t and no cost is charged for period t in the objective function (8). For combination B, if $X_{it}=1$ and $X_{it-1}=0$, i.e., in location i in period t an opened facility exists and there is no opened facility in period $t-1$, then the facility is opened in period t ($X''_{it}=1$) and an opening cost F''_{it} and operating cost F_{it} must be charged in this period. Also, it is observed for combination C that if $X_{it}=0$ and $X_{it-1}=1$, that is, if in the location i in period t there is not an opened facility and in period $t-1$ there is, it means that the facility is closed in period t and a closing cost F'_{it} must be charged in this period. Consequently, for the last combination D, if $X_{it}=1$ and $X_{it-1}=1$, i.e., if in the location i in period t there is an opened facility and there is also an opened facility in period $t-1$, it means that the facility continues operating from period $t-1$ to period t and only an operation cost F_{it} must be charged in this last period.

Additionally, constraints (13) ensure that stock capacity for each DC and product is satisfied with a probability of at least $1-\beta$. The constraints (14) state that ordering capacity is respected by the order size for product k at DC $i$at period $t$with a probability of at least $1-\gamma$. Equations (15) and (16) compute the mean and variance of the fulfilled demand by each warehouse $i$for product k at period t. Equation (17) calculates undershoot mean by each warehouse i for product

k at period t. Equation (18) computes undershoot variance by each warehouse i for product k at period t. Finally, (19) and (20) define decision variables domains.

IV. SOLUTION APPROACH

We know that the single product LIP is already an NP-hard problem. Accordingly, DMLIP is also an NP-hard problem, because it includes a higher complexity by incorporating multiple periods and products. In addition, the nonlinearity presents stochastic capacity constraints and objective function, making solving by commercial software prohibitive due to the high computational times involved. We tackle this issue by solving the model by employing the Lagrangian relaxation and subgradient algorithm. We relax constraints (15) and (16), i.e., demand's mean and variance of each product in every distribution center. In addition, we relax constraints (9), regarding customer allocation, like various Lagrangian relaxation implementations for conventional facility location problems and LIP. Then, we break down subproblems by separating binary variables of the supply chain network design (*X, X', X", W*, and *Y*) from ordering quantity decisions (*Q*). Finally, we solve exactly each warehouse's relaxed subproblems with a dynamic programming approach.

A. RELAXED PROBLEM AND DECOMPOSITION

By assigning the Lagrangian multipliers vectors ψ to constraint (9), and λ and ω to the set of constraints (15) and (16), respectively, we derive the incoming relaxed problem LR_{RP}:

$$\begin{aligned}\underline{LR_{RP}|}\ Min\ &\sum_{i=1}^{N}\sum_{t=1}^{T}\left(F_{it}X_{it}+F'_{it}X'_{it}+F''_{it}X''_{it}\right)\\&+\sum_{i=1}^{N}\sum_{k=1}^{K}\sum_{t=1}^{T}G_{ikt}W_{ikt}\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{k=1}^{K}\sum_{t=1}^{T}\\&+((RC_{ik}+\lambda_{ikt})d_{jkt}+TC_{ijt}+\omega_{ikt}v_{ikt}-\psi_{jkt})Y_{ijkt}\\&+\sum_{i=1}^{N}\sum_{k=1}^{K}\sum_{t=1}^{T}\left(OC_{ikt}\frac{D_{ikt}}{(Q_{ikt}+US_{ikt})}+HC_{ikt}\frac{(Q_{ikt}+US_{ikt})}{2}\right)\\&+\sum_{i=1}^{N}\sum_{k=1}^{K}\sum_{t=1}^{T}HC_{ikt}(Z_{1-\alpha}\sqrt{V(US_{ikt})+V_{ikt}LT_{ik}})\\&-\sum_{i=1}^{N}\sum_{k=1}^{K}\sum_{t=1}^{T}(\lambda_{ikt}D_{ikt}+\omega_{ikt}V_{ikt})+\sum_{j=1}^{M}\sum_{k=1}^{K}\sum_{t=1}^{T}\psi_{jkt}\\&\text{subject to: (10)}-\text{(14)},\ \text{(17)}-\text{(20)}\end{aligned}\tag{21}$$

With fixed values assigned to the Lagrangian multipliers, λ, ω and ψ we minimize (21) over operating and locating variables, $X_{it}, X'_{it}, X''_{it}$, W_{ikt}, and the customer assignment variables Y_{ijkt}. Given the dual variables vector λ, ω and ψ, the problem separates into the following subproblem for each distribution center i:

$$\begin{aligned}\underline{SP_i|}\ Min\ &\sum_{t=1}^{T}\left(F_{it}X_{it}+F'_{it}X'_{it}+F''_{it}X''_{it}\right)+\sum_{k=1}^{K}\sum_{t=1}^{T}G_{ikt}W_{ikt}\\&+\sum_{j=1}^{M}\sum_{k=1}^{K}\sum_{t=1}^{T}((RC_{ikt}+\lambda_{ikt})d_{jkt}+TC_{ijt}+\omega_{ikt}v_{ikt}-\psi_{jkt})Y_{ijkt}\\&+\sum_{k=1}^{K}\sum_{t=1}^{T}\left(OC_{ikt}\frac{D_{ikt}}{(Q_{ikt}+US_{ikt})}+HC_{ik}\frac{(Q_{ikt}+US_{ikt})}{2}\right)\\&+\sum_{k=1}^{K}\sum_{t=1}^{T}HC_{ikt}(Z_{1-\alpha}\sqrt{V(US_{ikt})+V_{ikt}LT_{ik}})\\&-\sum_{k=1}^{K}\sum_{t=1}^{T}(\lambda_{ikt}D_{ikt}+\omega_{ikt}V_{ikt})\\&\text{subject to: (10)}-\text{(14)},\ \text{(17)}-\text{(20)}\end{aligned}\tag{22}$$

Note for fixed values of the Lagrangian multipliers, the last term of (21), associated to ψ is a constant term, so we can remove it from (22) because it has no impact on the optimal value of subproblem SP_i. Thus, the further decomposition can be developed more easily. However, this term must be added in expression (37), once optimal values of subproblems have been obtained, because it was already part of the Lagrangian function (21).

We incorporate valid inequalities set $(D_{ikt}, V_{ikt}) \in \Omega_{kt}$ to address the above subproblems. These valid inequalities correspond to constraints that encompass the set of feasible solutions of the variables D_{ikt} and V_{ikt} [22]. With the Lagrangian multipliers held constant for each iteration, λ_{ikt}, ω_{ikt}, ψ_{jkt}, subproblems (22) were separated into an inventory subproblem SP^1_{ikt} and in a location-allocation subproblem SP^2_i as follows:

$$\begin{aligned}&\underline{SP^1_{ikt}|}\ \Pi_{ikt}\\&=Min\left(OC_{ikt}\frac{D_{ikt}}{(Q_{ikt}+US_{ikt})}+HC_{ikt}\frac{(Q_{ikt}+US_{ikt})}{2}\right)\\&\quad+HC_{ikt}(Z_{1-\alpha}\sqrt{V(US_{ikt})+V_{ikt}LT_{ikt}})\\&\quad-(\lambda_{ikt}D_{ikt}+\omega_{ikt}V_{ikt})\\&\text{s.t. :}\\&\quad Q_{ikt}+US_{ikt}+Z_{1-\alpha}\sqrt{V(US_{ikt})+V_{ikt}LT_{ikt}}\\&\quad+Z_{1-\beta}\sqrt{V_{ikt}LT_{ikt}}\le ICap_{ikt}\\&\quad Q_{ikt}+US_{ikt}+Z_{1-\gamma}\sqrt{V(US_{ikt})}\le QCap_{ikt}\\&\quad US_{ikt}=\frac{V_{ikt}}{2D_{ikt}}+\frac{D_{ikt}R_{ikt}}{2}\\&\quad V\left(US_{ikt}\right)=\frac{D^2_{ikt}R^2_{ikt}}{12}+\frac{V_{ikt}R_{ikt}}{2}-\frac{V^2_{ikt}}{4D^2_{ikt}}\\&\quad(D_{ikt},V_{ikt})\in\Omega_{kt}\end{aligned}\tag{23}$$

$$\begin{aligned}&\underline{SP^2_i|}\\&Min\ H_i=\sum_{t=1}^{T}\left(F_{it}X_{it}+F'_{it}X'_{it}+F''_{it}X''_{it}\right)\\&\quad+\sum_{k=1}^{K}\sum_{t=1}^{T}(G_{ikt}+\Pi_{ikt})W_{ikt}\end{aligned}$$

$$+\sum_{j=1}^{M}\sum_{k=1}^{K}\sum_{t=1}^{T}((RC_{ikt}+\lambda_{ikt})d_{jkt}+TC_{ijt}+\omega_{ikt}v_{ikt}-\psi_{jkt})Y_{ijkt} \tag{24}$$

s.t. :

$$Y_{ijkt} \le W_{ikt} \quad \forall j \in J, \forall k \in K, \forall t \in T \tag{25}$$

$$W_{ikt} \le X_{it} \quad \forall k \in K, \forall t \in T \tag{26}$$

$$X_{it} - X_{it-1} = X''_{it} - X'_{it} \quad \forall t \in T \tag{27}$$

H_i is the benefit of opening DC i to the total cost espression (8). The breakdown involves solving SP^1_{ikt} to calculate Π_{ikt}, and then solving SP^2_i to compute H_i, according to calculated Π_{ikt}, as detailed in subsection IV-C.

B. INVENTORY SUBPROBLEMS SOLVING

With predetermined values set for the Lagrangian multipliers λ_{ikt}, ω_{ikt} and ψ_{jkt} Associated with relaxed constraints (15), (16), and (9), the fixed Lagrangian multipliers yield a non-feasible solution for the original problem in every algorithm's iteration. However, this result produces a lower bound of the optimal solution for the primal problem. So, we resolve SP^1_{ik} to compute Π_{ikt} in expressions (23).

To solve SP^1_{ik}, the optimal order will be calculated as follows:

$$Q^*_{ikt} = \max\left\{\min\left\{Q^{EOQ}_{ikt},\ Q'_{ikt},\ Q''_{ikt}\right\}, 0\right\}$$

where:

$$Q^{EOQ}_{ikt} = \sqrt{(2OC_{ik}D_{ik})/HC_{ik}} - US_{ik}$$

$$Q'_{ikt} = ICap_{ikt} - US_{ikt} - Z_{1-\alpha}\sqrt{V(US_{ikt}) + V_{ikt}LT_{ikt}} - Z_{1-\beta}\sqrt{V_{ikt}LT_{ikt}}$$

$$Q''_{ikt} = QCap_{ikt} - US_{ikt} - Z_{1-\gamma}\sqrt{V(US_{ikt})}$$

Once Π_{ikt} is achieved, SP^2_i is resolved according to the value of Π_{ikt}.

C. AN EXACT SOLUTION FOR SUBPROBLEMS BY USING DYNAMIC PROGRAMMING

In this section we introduce a new exact method to solve the subproblems using dynamic programming. To visualize the dynamic structure of the model into objective function we isolate the variable X''_{it} from constraint (27), $X''_{it} = X_{it} - X_{it-1} + X'_{it}$ and replacing in (24). So, the subproblems SP^2_i can be written as follows:

$$Min\ H_i = \sum_{t=1}^{T}\left(\begin{array}{l}(F_{it} + F''_{it})X_{it} - F''_{it}X_{it-1} + (F'_{it} + F''_{it})X'_{it} \\ + \sum_{k=1}^{K}\left((G_{ikt} + \Pi_{ikt})W_{ikt} + \sum_{j=1}^{M}\delta_{ijkt}Y_{ijkt}\right)\end{array}\right)$$

subject to: (25), (26) and (27)

$$\text{where } \delta_{ijkt} = ((RC_{ik} + \lambda_{ikt})d_{jkt} + TC_{ij} + \omega_{ikt}v_{ikt} - \psi_{jkt}) \tag{28}$$

Notwithstanding the previous substitution in the objective function, the constraints (27) cannot be taken out of (28) because we must hold the logic relations and combinations between operation variables, $X_{it} - X_{it-1}$ and opening and closing variables, $X''_{it} - X'_{it}$, as shown in Table 4. An alternative to eliminate these constraints (27) from the model is to incorporate a mixed quadratic program formulation into the objective function.

We solved efficiently and exactly each subproblem (28) by using dynamic programming. We denoted by $H_{i\tau}$ the expression of H_i for the τ first time periods of the dynamic programming, by $H_{i\tau}(1)$ the minimum value of $H_{i\tau}$ if $X_{i\tau} = 1$ and by $H_{i\tau}(0)$ the minimum value of $H_{i\tau}$ if $X_{i\tau} = 0$. Thus, for $\tau = 1$ we obtain

$$H_{i1} = (F_{i1} + F''_{i1})X_{i1} - F''_{i1}X_{i0} + (F'_{i1} + F''_{i1})X'_{i1} + \sum_{k=1}^{K}(G_{ik1}+\Pi_{ik1})W_{ik1} + \sum_{j=1}^{M}\sum_{k=1}^{K}\delta_{ijk1}Y_{ijk1} \tag{29}$$

$$H_{i1}(0) = 0 \tag{30}$$

$$H_{i1}(1) = F_{i1} + F''_{i1} + \sum_{k=1}^{K}\min\left(\left((G_{ik1}+\Pi_{ik1}) + \sum_{j=1}^{M}\min(\delta_{ijk1}, 0)\right), 0\right) \tag{31}$$

And for $\tau > 1$:

$$H_{i\tau} = \sum_{t=1}^{\tau}\left(\begin{array}{l}(F_{it} + F''_{it})X_{it} - F''_{it}X_{it-1} + (F'_{it} + F''_{it})X'_{it} \\ + \sum_{k=1}^{K}\left((G_{ikt} + \Pi_{ikt})W_{ikt} + \sum_{j=1}^{M}\delta_{ijkt}Y_{ijkt}\right)\end{array}\right) \tag{32}$$

$$= H_{i\tau-1} + (F_{i\tau}+F''_{i\tau})X_{i\tau} - F''_{i\tau}X_{i\tau-1} + (F'_{i\tau} + F''_{i\tau})X'_{i\tau} + \sum_{k=1}^{K}\left((G_{ik\tau} + \Pi_{ik\tau})W_{ik\tau} + \sum_{j=1}^{M}\delta_{ijk\tau}Y_{ijk\tau}\right), \quad \text{so that} \tag{33}$$

$$H_{i\tau}(0) = \min\left(H_{i\tau-1}(0), H_{i\tau-1}(1) + F'_{i\tau}\right) \tag{34}$$

$$H_{i\tau}(1) = \min\left(H_{i\tau-1}(0) + F_{i\tau} + F''_{i\tau},\ H_{i\tau-1}(1) + F_{i\tau}\right) + \sum_{k=1}^{K}\min(\Delta_{ik\tau}, 0)$$

where $\Delta_{ik\tau}$

$$= (G_{ik\tau} + \Pi_{ik\tau}) + \sum_{j=1}^{M}\min(\delta_{ijk\tau}, 0) \tag{35}$$

Note in (34) that if $X_{i\tau} = 0$, then $H_{i\tau} = H_{i\tau-1}$, for $X_{i\tau-1} = 0$. This is, if in the location i in period τ there does not exist an opened facility and also in period $\tau - 1$, no cost is charged and the total cost in period τ is the same of the total cost in period $\tau - 1$. On the other hand, if $X_{i\tau} = 0$, then $H_{i\tau} = H_{i\tau-1} + F'_{i\tau}$ for $X_{i\tau-1} = 1$. Thus, in the location i in period τ there does not exist an opened facility and there does exist an opened facility in period $\tau - 1$. It means that the facility is closed in period τ and a closing cost $F'_{i\tau}$ must be charged in this period. In addition, it observes in (35) that if $X_{i\tau} = 1$, then $H_{i\tau} = H_{i\tau-1} + F_{i\tau} + F''_{i\tau}$, for $X_{i\tau-1} = 0$. This is, if in the location i in period τ there exists an opened facility and in period $\tau - 1$ not, means that the facility is opened in period τ and an opening cost $F''_{i\tau}$ and operating cost $F_{i\tau}$ must be charged in this period. Consequently, if $X_{i\tau} = 1$, then $H_{i\tau} = H_{i\tau-1} + F_{i\tau}$ for $X_{i\tau-1} = 1$. So, in the location i in period τ there exists an opened facility and there exists an opened facility in period $\tau - 1$. It means that the facility continues operating from period $\tau - 1$ to period τ and only an operation cost $F_{i\tau}$ must be charged in this last period.

Finally,

$$H_i^* = H_{iT}^* = \min\left(H_{iT}(0), H_{iT}(1)\right) \tag{36}$$

Therefore, the optimal value of the Lagrangian function is equal to the sum, for all $i = 1, \ldots, N$, of the optimal values of the subproblems solved by dynamic programming plus the constant term of the Lagrange multipliers as follows:

$$LR_{RP}^* = \sum_{i=1}^{N} H_i^* + \sum_{j=1}^{M}\sum_{k=1}^{K}\sum_{t=1}^{T} \psi_{jkt} \tag{37}$$

The second term of (37) is the same constant term of the Lagrangian function (21), which is not considered to minimize it and therefore we now must add it. The expression (37) provides a lower bound with respect to the optimal value of the original or primal problem. Because we need to find the optimum values of the decisional variables of the subproblems to apply the subgradient method, we work backwards, and we first obtain the optimal values of the operating variable for the last period T as:

$$X_{iT} = \begin{cases} 0 & \text{if } H_{iT}^* = H_{iT}(0) \\ 1 & \text{otherwise} \end{cases} \tag{38}$$

Then, we find the values for the operating variables for the rest of periods $T - 1, \ldots, 1$ as follows:

$$X_{it} = \begin{cases} 0 & \text{if } X_{it+1} = 0 \text{ and } H_{it+1}(0) = H_{it}(0) \\ 1 & \text{if } X_{it+1} = 0 \text{ and } H_{it+1}(0) = H_{it}(1) + F'_{it+1} \\ 0 & \text{if } X_{it+1} = 1 \text{ and } H_{it+1}(1) = H_{it}(0) + F_{it+1} \\ & \quad + F''_{it+1} + \Delta_{ikt+1} \\ 1 & \text{if } X_{it+1} = 1 \text{ and } H_{it+1}(1) = H_{it}(1) + F_{it+1} \\ & \quad + \Delta_{ikt+1} \end{cases} \tag{39}$$

Once we find all the values of X_{it}, we can obtain the values of the closing and opening variables X'_{it} and X''_{it} as:

$$X'_{it} = \begin{cases} 1 & \text{if } X_{it} = 0 \text{ and } X_{it-1} = 1 \\ 0 & \text{if not} \end{cases} \tag{40}$$

$$X''_{it} = \begin{cases} 1 & \text{if } X_{it} = 1 \text{ and } X_{it-1} = 0 \\ 0 & \text{if not} \end{cases} \tag{41}$$

Then, we obtain the values of the assignment variables W_{ikt} and Y_{ijkt} as:

$$W_{ikt} = \begin{cases} 1 & \text{if } X_{it} = 1 \text{ and } \Delta_{ikt} < 0 \\ 0 & \text{if not} \end{cases} \tag{42}$$

$$Y_{ijkt} = \begin{cases} 1 & \text{if } W_{ikt} = 1 \text{ and } \delta_{ijkt} < 0 \\ 0 & \text{if not} \end{cases} \tag{43}$$

Finally, we find the optimal values of variables of the subproblems Π_{ikt} according to the following:

If $W_{ikt} = 1$ then $D_{ikt}, V_{ikt}, US_{ikt}, V(US_{ikt})$ and Q_{ikt} hold the values calculated from the solving subproblems (23). If $W_{ikt} = 0$ then $D_{ikt} = V_{ikt} = US_{ikt} = V(US_{ikt}) = Q_{ikt} = 0$, $Y_{ijkt} = 0$, $\forall j = 1, \ldots, M$.

D. OBTAINING A FEASIBLE SOLUTION

We suggest a Lagrangian heuristic to obtain a feasible solution during each iteration r by using the information of the last lower bound result obtained for the Lagrangian algorithm, i.e., through expression (37). This feasible solution is an upper bound to the primal problem´s optimal value. To decrease the difference between the upper and lower bound (gap), it is imperative to design a heuristic that can obtain good feasible solutions. Therefore, we structure the Lagrangian heuristic based on three major algorithms: selection of distribution centers, customers' greedy assignment, and K-OPT local search improvements. These procedures were executed for varying numbers of distribution centers, ranging from 1 to N, each product from 1 to K and each period from 1 to T. In other words, the entire algorithm is run $N \cdot K \cdot T$ times. So, the best solution, i.e., the lower value, is selected as a feasible solution of the iteration r. The above is founded on dual information and solutions to the subproblems SP_i. We emphasize the considerable complexity of the heuristic, particularly the local search enhancement procedures, in contrast to the standard basic Lagrangian heuristic found in the literature. To mitigate the risk of long CPU times, we only execute the K-OPT subroutine for each 30 iterations of the entire Lagrangian relaxation algorithm [9]. The three main algorithms are developed according to the following:

1) WAREHOUSE SELECTION

This routine considers that the Lagrange multipliers $(\lambda_{ikt}^r, \omega_{ikt}^r, \psi_{ikt}^r)$ and the optimal solution $\bar{x}^r = (\bar{X}^r, \bar{X}'^r, \bar{X}''^r, \bar{W}^r, \bar{Y}^r, \bar{D}^r, \bar{V}^r, \bar{Q}^r)$ of the subproblems SP_i are known. For the warehouse selection, the optimal costs of subproblems SP_i are taken as initial values. Then, the best P warehouses are chosen, assuring the fulfillment of the capacity constraints. The warehouse selection algorithm follows these steps:

Step 1: Utilize the results obtained from Algorithm 1: $\bar{x}^r, \lambda_{ikt}^r, \omega_{ikt}^r, \psi_{ikt}^r$.

Step 2: Compute $\bar{\Pi}_{ikt}$ using (23), for each DC i, product k, and time t.

Step 3: Calculate $\bar{H}_i$ using (24), for each DC i.

Step 4: Arrange $\bar{H}_i$ in increasing order.

Step 5: Choose P DCs in increasing order of $\bar{H}_i$.

2) GREEDY CUSTOMER ASSIGNMENT

Once warehouse selection is finished, the customers are allocated to the selected warehouses using a greedy approach. This assignment is based on the transportation cost $RC_{ikt}d_{jkt} + TC_{ij}$, so each customer is allocated to the closest warehouse to fill the ordering and inventory constraints. We calculate three types of order quantities for each warehouse i, product $k$and period t, to satisfy these constraints. First, Q'_{ikt} which represents the available inventory capacity after the undershoot, safety stock and inventory associated with variance are subtracted from $ICap$, i.e., inventory capacity. Second, Q''_{ikt}which isthe available order quantity after subtracting the undershoot and undershoot variances from the order capacity ($QCap$). Third, Q^{EOQ}, which corresponds to the economic order quantity without considering inventory and ordering capacity.

3) LOCAL SEARCH IMPROVEMENTS

Once feasible solution $\tilde{x}^r = (\tilde{X}^r, \tilde{X}'^r, \tilde{X}''^r, \tilde{W}^r, \tilde{Y}^r, \tilde{D}^r, \tilde{V}^r, \tilde{Q}^r)$ is obtained for each iteration r, following the completion of the selection of distribution centers and customers' greedy assignment, the two local search improvements are executed. The first improvement, called 1-OPT, calculates the redistribution of each retailer, product, and period to other opened warehouses while adhering to capacity limitations. First, the minimum upper bound interchange is selected, and if the objective function decreases, the reassignment becomes final. The second improvement, known as 2-OPT, involves considering two retailers in distinct DCs and exchanging them while maintaining capacity constraints. The exchange converts permanently if the objective function decreases due to the swap. In this proposed Lagrangian relaxation approach, the minimum value of the dual problem is determined through dual optimization, which serves as a lower bound for the optimal quantity of the primal problem. Consequently, Lagrangian heuristic provides the difference between the cost of the best feasible solution obtained and this lower bound, which is an upper bound for the heuristic solutions' errors.

E. SUBGRADIENT METHOD

When the subproblems have already been solved, updating the dual variables values in every iteration r is necessary. This update aims to enhance the lower bound using the subgradient method, as [23] and [24] illustrated. This method utilizes the violation/slackness vector related to relaxed constraints in the following manner:

$$SD^r_{ikt} = \left(\sum_{j=1}^{M} Y_{ijkt} d_{jkt}\right) - D_{ik} \quad \forall i, \forall k, \forall t$$

$$SV^r_{ikt} = \left(\sum_{j=1}^{M} Y_{ijkt} v_{jkt}\right) - V_{ikt} \quad \forall i, \forall k, \forall t$$

$$SY^r_{jkt} = 1 - \left(\sum_{i=1}^{N} Y_{ijkt}\right) \quad \forall j, \forall k, \forall t$$

For each iteration r, the step sizes are computed by following expressions:

$$t^r_\lambda = \alpha^r \frac{(UB - LB_r)}{\sum_{ikt} (SD^r_{ikt})^2}$$

$$t^r_\omega = \alpha^r \frac{(UB - LB_r)}{\sum_{ikt} (SV^r_{ikt})^2}$$

$$t^r_\psi = \alpha^r \frac{(UB - LB_r)}{\sum_{jkt} (SY^r_{jkt})^2}$$

UB represents the minimum upper bound on the primal problem's optimal solution up to iteration r, achieved by executing the Lagrangian heuristic algorithm discussed earlier. LB_r indicates the lower bound resulting from solving subproblems during iteration r. Also, α^r corresponds to the scale factor and usually meets the condition $0 \leq \alpha^r \leq 2$. Thus, the dual variables or Lagrangian multipliers are updated through the following equations:

$$\lambda^{r+1}_{ikt} = \max\left\{\lambda^r_{ikt} + t^r_\lambda SD^r_{ikt},\ 0\right\};$$

$$\omega^{r+1}_{ikt} = \max\left\{\omega^r_{ikt} + t^r_\omega SV^r_{ikt},\ 0\right\} \quad \text{and}$$

$$\psi^{r+1}_{jkt} = \max\left\{\psi^r_{jkt} + t^r_\psi SY^r_{jkt},\ 0\right\}$$

The process continues up to a termination condition is satisfied.

V. COMPUTATIONAL RESULTS

In this section, we define computational experiments and evaluate the quality of the feasible solutions obtained by the proposed solution approach. In addition, we validate the DMLIP model and its heuristic solutions through sensitivity analysis. The numerical experiments consider two different data sets. Both sets consider five commodities and a five-year planning horizon. The first set comprises 20 warehouses and 20 customers, and the second one comprises 20 DCs and 40 clients. We define the cost parameters based on a general cost unit, called CU. To simplify, we suppose identical ordering costs (OC), lead times (LT), and fixed costs (G) for all products within each DC. The data is the same as used in Araya-Sassi et al. (2020). It assumes a value of 100 CU for transportation costs, RC and holding costs, HC. Additionally, $ICap$is set to 1200. Finally, the $Z_{1-\alpha}$, $Z_{1-\beta}$ and $Z_{1-\gamma}$ are all equal to 1.64, representing a 95% of service level.

We rigorously tested the proposed Lagrangian relaxation algorithm for the DMLIP in 36 instances. These experiments, presented in a comparative manner, demonstrate algorithm solutions quality by measuring their gaps concerning the optimal values of the relaxed problem. As mentioned before, the optimal solution of the dual problem serves as a lower bound for the optimal solution for the primal problem. We developed a sensitivity analysis of the main parameters, including demand variability and fixed costs, to validate the model

TABLE 5. Results for the first data set: 20 DCs x 20 Customers, R=1 and QCap=1200.

Prob no.	FC	FV	DCs opened (*)	DCs closed (*)	No. of open DCs	UB	LB	% Gap	Lag iter	CPU time (s)
1	0.7	0.7	None	None	3	28,453,255	28,143,340	1.10	1,594	1,000
2	0.7	1.0	None	None	3	28,705,478	28,470,129	0.83	1,790	1,129
3	0.7	1.3	None	None	3	28,978,270	28,799,059	0.62	1,962	1,500
4	1.0	0.7	None	None	3	30,653,334	30,343,454	1.02	1,704	1,005
5	1.0	1.0	2,11,12	N/A	3	30,905,557	30,669,884	0.77	1,622	978
6	1.0	1.3	None	None	3	31,178,349	30,999,146	0.58	1,997	1,248
7	1.3	0.7	None	None	3	32,853,413	32,381,838	1.46	3,961	2,431
8	1.3	1.0	None	None	3	33,105,636	32,753,869	1.07	1,133	686
9	1.3	1.3	None	None	3	33,378,429	33,058,209	0.97	1,105	660

(*) DCs opened and closed during planning horizon compared to base case, Prob no 5; N/A, not applicable.

TABLE 6. Results for the first data set: 20 DCs x 20 Customers, R=3 and QCap=1200.

Prob no.	FC	FV	DCs opened (*)	DCs closed (*)	No. of open DCs	UB	LB	% Gap	Lag iter	CPU time (s)
10	0.7	0.7	11,14	3,12	7	35,675,072	34,080,931	4.68	2,154	1,155
11	0.7	1.0	11,14	3,12	7	36,444,198	34,654,543	5.16	2,349	1,261
12	0.7	1.3	11	8	7	37,053,288	35,522,604	4.31	2,271	1,189
13	1.0	0.7	11	3,12	6	39,093,539	37,660,411	3.81	2,259	1,155
14	1.0	1.0	1,2,3,5,8,10,12	N/A	7	40,674,524	38,284,732	6.24	2,320	1,198
15	1.0	1.3	13,20	1,3	7	41,614,523	39,397,978	5.63	2,553	1,336
16	1.3	0.7	11	3,12	6	42,834,881	41,073,397	4.29	2,661	1,371
17	1.3	1.0	4,14	10,12	7	44,589,188	41,806,613	6.66	2,626	1,383
18	1.3	1.3	None	None	7	45,671,537	43,162,544	5.81	2,275	1,196

(*) DCs opened and closed during planning horizon compared to base case, Prob no 14; N/A, not applicable.

and the proposed heuristic. We considered only one order capacity level, i.e., *QCap*=1200. Warehouse demand variances and fixed location costs varied by three values from the base case. We examined the review periods $R = 1, 3$. The average execution time for Lagrangian relaxation data sets was 1216 and 7499 seconds, respectively. The complete heuristic algorithm was run on a system featuring an Intel i7 processor at 2.4 GHz, including a RAM of 16 GB, and executing on the version 10 of Windows platform. Microsoft Visual Studio 2017 C++ was used to write the code.

The parameters utilized in the Lagrangian relaxation approach for all instances. First, the maximum iterations number was set to 5000 [9], taking this quantity into account as a sensible threshold wherein the algorithm can make enhancements. Next, the starting scale factor α was established at 2, with the lower value of α set to $1 \cdot 10^{-7}$.

We opted for these values because, from experience, they have yielded satisfactory outcomes [9]. Additionally, the decision to halve αafter 30 iterations was made based on [9].

Subsequently, the dual variables were initially set to 0, which is the conventional practice; nonetheless, in certain instances, it could be enhanced. Finally, the termination condition for an optimality gap of $1 \cdot 10^{-3}$% was established following [25]. Previous quantities underwent empirical testing to determine the settings that yielded the best convergence. The expression *(UB-LB)/LB)* $\times$*100*is utilized to compute the optimality gap.

The details of the solutions of the DMLIP model are exposed in Table 5 and Table 6 for the first data set, Table 7 and Table 8 for the second one, in which the columns correspond to the following: *Prob no.:*Problem number, *FC:*Fixed cost sensitivity factor (i.e., 1.3 refers to a 30% increase), *FV*: Variance sensitivity factor, *DCs closed:*The further distribution centers (DCs) that were closed comparatively to the baseline case, *DCs opened:*The further distribution centers (DCs) that were opened comparatively to the baseline case, *No. of open DCs:*Total number of open DCs, *UB:*Upper Bound or objective function value of the best feasible solution, *LB*: the best Lower Bound found for optimal objective function, *% Gap:*Gap percentage between lower bound and upper bound solution, *Lag iter:*Total number of Lagrangian algorithm iterations and *CPU time (s):*The amount of CPU time in seconds elapsed upon algorithm's termination. Tables 5 to 8 provide an overview of the outcomes of several combinations of significant parameter variations.

Note that upper bound values become greater when the demand variance (*FV*) increases. A comparable outcome is noted for changes in fixed cost (*FC*). Therefore, the Lagrangian heuristic denotes a reasonable response since system costs are expected to rise for both parameters' data sets. On the other hand, if we contrast results for $R = 1$ and $R = 3$ for both data sets, an increase in the review period's length ($R = 1$ to $R = 3$) is the worst solution in terms of the % gap and total cost. These results demonstrate the Lagrangian

TABLE 7. Results for second data set: 20 DCs x 40 Customers, R=1 and QCap=1200.

Prob no.	FC	FV	DCs opened (*)	DCs closed (*)	No. of open DCs	UB	LB	% Gap	Lag iter	CPU time (s)
19	0.7	0.7	11	8	5	47,670,892	47,131,930	1.14	2,260	7,635
20	0.7	1.0	11,14	1,8	5	48,773,622	48,096,766	1.41	1,865	6,264
21	0.7	1.3	14	None	6	49,717,691	49,011,840	1.44	2,239	7,239
22	1.0	0.7	None	None	5	50,821,424	50,185,496	1.27	1,686	5,742
23	1.0	1.0	1,2,3,5,8	N/A	5	52,230,129	51,231,169	1.95	2,041	10,218
24	1.0	1.3	14	None	6	53,421,430	52,330,523	2.08	1,788	5,879
25	1.3	0.7	13	5	5	53,744,352	53,109,039	1.20	1,901	6,771
26	1.3	1.0	None	None	5	55,386,523	54,302,999	2.00	2,143	7,854
27	1.3	1.3	14	None	6	57,125,170	55,551,195	2.83	1,915	6,860

(*) DCs opened and closed during planning horizon compared to base case, Prob no 23; N/A, not applicable.

TABLE 8. Results for the second data set: 20 DCs x 40 Customers, R=3 and QCap=1200.

Prob no.	FC	FV	DCs opened (*)	DCs closed (*)	No. of open DCs	UB	LB	% Gap	Lag iter	CPU time (s)
28	0.7	0.7	19	15	13	63796135	61239424	4.17	3107	6436
29	0.7	1.0	19	15	13	64863159	62379565	3.98	3883	8654
30	0.7	1.3	6,19	1,15	13	66258258	63484468	4.37	3515	7031
31	1.0	0.7	None	7	12	70955008	68251884	3.96	2667	5988
32	1.0	1.0	1,2,3,5,7,8,10,12,13,14,15,16,20	N/A	13	73137857	69602691	5.08	3478	7655
33	1.0	1.3	14	None	14	75301353	71054526	5.98	5000	9976
34	1.3	0.7	None	None	13	78531779	75230486	4.39	3052	7054
35	1.3	1.0	None	None	13	81201312	76799975	5.73	3503	8056
36	1.3	1.3	17,19	1,15	13	83537482	78553275	6.35	5000	9675

(*) DCs opened and closed during planning horizon compared to base case, Prob no 32; N/A, not applicable.

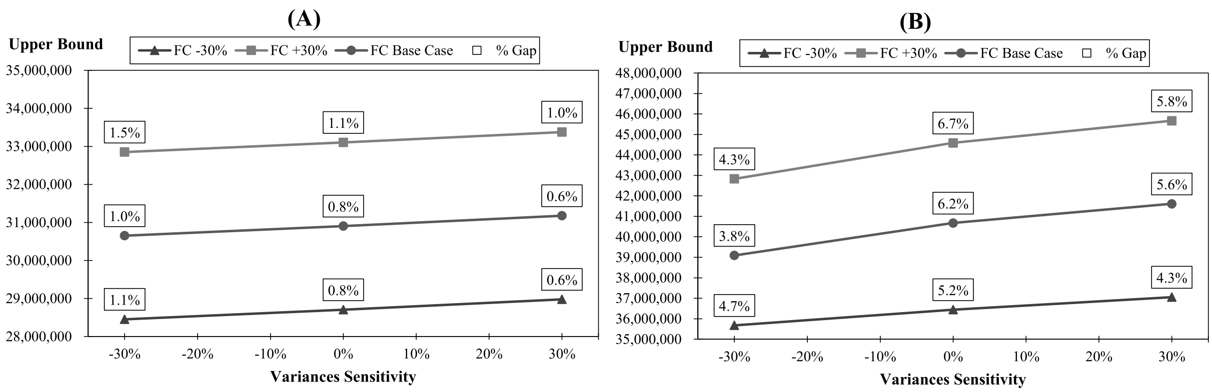

FIGURE 3. % gap and UB for fixed cost and variances sensitivity for first data set with R=1;3 and QCap=1200.

heuristic's effectiveness according to the objective function's tendencies when various input parameters are adjusted.

As the review interval is extended from 1 to 3 days, the quantity of distribution centers opened increases from 3 to 7 for the first dataset (see Tables 5 and 6) and from 5 to 13 for the second one (see Tables 7 and 8). In addition, the CPU times significantly increase when we compare both data sets.

The average computer time for the first data set is 1216 seconds or 20 minutes, while the second data set is 7500 seconds or nearly two hours. This demonstrates that DMILP is an NP-hard problem in which computational time increases in a non-polynomial form as we double the number of customers in the model.

Tables 7 and 8 highlight that the computational time obtained near two hours is reasonable for the second data set, i.e., 20 DCs and 40 customers, which is a medium-scale instance. The reason is that, in general, the network design activity is a strategic task to support strategic planning decisions, and therefore, it is developed every certain number of years.

Finally, we observe a relevant tradeoff between the review period and total system cost. For the second dataset, when

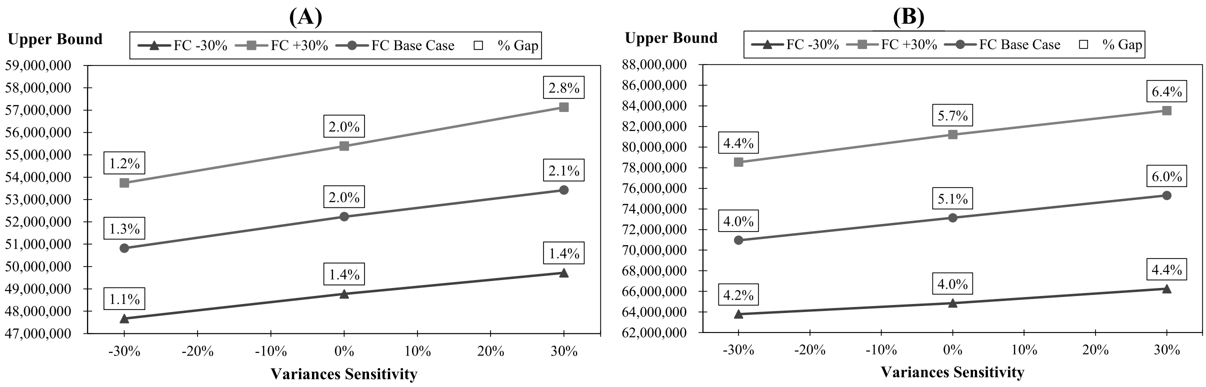

FIGURE 4. % gap and UB for fixed cost and variances sensitivity for second data set with R=1;3 and QCap=1200.

it increases from $R = 1$ to $R = 3$, the average rise in the objective function is approximately 40%. The previous shows the significant impact that generates the review period selection decision on the inventory control policy and, therefore, facility location decisions.

Fig. 3 and 4 depict the objective function values of the model's heuristic solutions applied to the first and second dataset, consisting of 20 distribution centers (DCs) serving 20 and 40 customers, respectively, and the corresponding % gap between the upper and lower bounds. In Fig. 3A and 3B, the sensitivity to variances and fixed costs is illustrated, assuming an ordering capacity of 1200 for review periods of one and three days, respectively. Fig. 3A and 3B, show the values across three variance values for each of the three fixed cost levels: −30%, +30%, and the baseline case. The upper bound becomes more significant with fixed cost and variance growth.

Analogously, Fig. 4A and 4B depict the variance and fixed cost sensitivity, considering an order capacity of 1200, for reviewing periods of one and three days, respectively. It is noticeable that total cost increases as both fixed cost and variance are heightened.

VI. CONCLUSION AND FUTURE RESEARCH

This article introduces a dynamic multi-product location-inventory model featuring a periodic review inventory policy and the modularity of stochastic capacity constraints, considering risk pooling impacts. Unlike conventional single-product integer linear programming (LIP), this model realistically extends to multiple products and periods, incorporating modular stochastic capacity constraints. Its design permits for reduced fixed costs by determining the precise required capacity of product categories over the planning horizon for each distribution center rather than allocating all capacity, which may lead to suboptimal solutions. The dynamic multi-product integer linear programming (DMLIP) model shares similarities with LIP in multi-product scenarios and the facility location problem. However, the fixed cost component in the objective function is changed to reflect the optimal total fixed cost of modular capacities opening in DCs for every product instead of the overall capacity seen in conventional facility location problem (FLP) models. In addition, the modularity of stochastic capacity constraints operates separately for each DC product, enabling a decomposition approach to handle large-scale data in business applications effectively.

Furthermore, the proposed Lagrangian relaxation approach, which includes a novel dynamic programming algorithm to solve the subproblems, has been demonstrated to successfully address the presented DMLIP with modular stochastic capacity constraints model. So, this research represents an original and remarkable contribution to both academic research and industrial applications. We highlight incorporating valid inequalities and 1-Opt and 2-Opt algorithms to obtain near-optimal feasible solutions, the key to obtaining them at the same low gaps between the upper and lower bounds. These computational results involve that this solution approach could be applied in great-scale problem data and high complexity location-inventory problems such as dynamic multi-tier and multi-source formulations.

From a managerial perspective, this paper introduces a dynamic multi-product location-inventory model with a periodic review control policy. This model addresses current management needs regarding efficient network design, particularly in markets characterized by high uncertainty. Capacity modularity enables Supply Chain Network Design (SCND) strategies that incorporate mixed approaches, like utilizing public warehouse space rental alongside private warehouses, mainly when one distribution center serves only a subset of total products. This high adaptability is founded in capacity expansion and reduction of additional product categories, along with the temporary closure of distribution centers and the relocation of capacity modules. Depending on factors such as capital budgeting and product category level, managers can obtain invaluable insights to support their decision-making process. This decision is significant when we analyze the supply chain network configuration and, as in this case, when the review period of inventory control is relevant. Finally, when a high storage space utilization at a warehouse is produced by review periods, the design and performance of the strategic network are affected.

Additional research is recommended to address certain issues, such as simultaneously incorporating two relevant time periods, one tactical and another strategic. The first period is related to inventory control decisions (i.e., days), and the second is linked with facility location decisions (i.e., years). Thus, we encourage the study of dynamic stochastic programming to approach within-year short-term inventory decisions to integrate them with strategic facility location decisions. So, given the high complexity of solving the proposed stochastic structure of this model, we consider the simulation approach by commercial software (e.g., Arena Simulation software) another plausible line of research for this issue. The key to successful real-world industrial applications of the SCND is to satisfy the companies' current needs with realistic models and efficient algorithms. Different extensions may be addressed to achieve this and generalize the proposed modeling structure, such as multisource, i.e., multiple plants or central DCs, multi-echelon configurations, and incorporating other relevant inventory control strategies (e.g., continuous review policy). We are currently working on these last issues.

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