

Almost automorphic solutions of dynamic equations on time scales

Carlos Lizama^{a,*}, Jaqueline G. Mesquita^{b,2}

^a Universidad de Santiago de Chile, Facultad de Ciencia, Departamento de Matemática y Ciencia de la Computación, Casilla 307, Correo 2, Santiago, Chile

^b Universidade de São Paulo, Campus de São Carlos, Instituto de Ciências Matemáticas e de Computação, Caixa Postal 668, 13560-970 São Carlos, SP, Brazil

Received 31 January 2013; accepted 18 June 2013

Available online 28 June 2013

Communicated by Daniel W. Stroock

Abstract

In the present work, we introduce the concept of almost automorphic functions on time scales and present the first results about their basic properties. Then, we study the nonautonomous dynamic equations on time scales given by $x^\Delta(t) = A(t)x(t) + f(t)$ and $x^\Delta(t) = A(t)x(t) + g(t, x(t))$, $t \in \mathbb{T}$ where \mathbb{T} is a special case of time scales that we define in this article. We prove a result ensuring the existence of an almost automorphic solution for both equations, assuming that the associated homogeneous equation of this system admits an exponential dichotomy. Also, assuming that the function g satisfies the global Lipschitz type condition, we prove the existence and uniqueness of an almost automorphic solution of the nonlinear dynamic equation on time scales. Further, we present some applications of our results for some new almost automorphic time scales. Finally, we present some interesting models in which our main results can be applied.

© 2013 Elsevier Inc. All rights reserved.

Keywords: Time scales; Almost automorphic; Nonautonomous dynamic equations; Exponential dichotomy

* Corresponding author.

E-mail addresses: carlos.lizama@usach.cl (C. Lizama), jaquebg@icmc.usp.br (J.G. Mesquita).

¹ Partially supported by project PIA-CONICYT ACT 1112: Red de Análisis Estocástico y Aplicaciones.

² Supported by CAPES grant 5811/12-0.

1. Introduction

The theory of time scales is a recent theory which started to be developed by Stefan Hilger, on his doctoral thesis (see [32]). This theory represents a powerful tool for applications to economics, populations models, quantum physics among others. See, for instance, [4,19,33]. Because of this fact, it has been attracting the attention of many mathematicians (see [1,6,10,22,24,23,26,27,43–46], for instance and the references therein) and the interest in the subject remains growing.

Since time scale is any closed nonempty subset of \mathbb{R} , the theory of dynamic equations on time scales allows to unify several developments in evolution equations, depending on the chosen time scale. For instance, if $\mathbb{T} = \mathbb{Z}$, we have a result for difference equations. On the other hand, taking $\mathbb{T} = \mathbb{R}$, we obtain a result for differential equations. We point out that this theory can also describe continuous-discrete hybrid processes, which have several important applications. For instance, the continuous-discrete hybrid processes can be used to investigate option-pricing and stock dynamics in finance, the frequency of markets and duration of market trading in economics, large-scale models of DNA dynamics, gene mutation fixation, electric circuits, populations models, among others. See, for instance, [13,19,34,48,33] and the references therein.

Moreover, this theory can be used to study quantum physics. Choosing the time scale equal to $q^{\mathbb{Z}} \cup \{0\}$, $q > 1$, we obtain a result for quantum calculus, which is a fundamental tool to study quantum physics. See [11] and [12] for more details.

Recently, the qualitative properties of the solutions of dynamic equations on time scales have been extensively investigated, specially concerning their periodicity. Periodic dynamic equations on time scales have been treated by several mathematicians. See, for instance, [1,2,6,10,39] and the references therein. On the other hand, almost periodicity is a recent concept in the literature of time scales. It was formally introduced by Y. Li and C. Wang (2011) in [37] and based on it, some results concerning almost periodicity for dynamic equations on time scales were proved (see [36]). However, to the best of our knowledge, the concept of almost automorphic functions on time scales has not been introduced in the literature until now.

The theory of continuous almost automorphic functions was introduced by S. Bochner in relation to some aspects of differential geometry (see [9,7,8]) and after that, this theory has been attracting the attention of several mathematicians and the interest in this topic still increases. See [16,14,15,17,20,29,30,40], for instance, and the references therein.

Motivated by this fact, the main goal of this paper is to introduce the concept of almost automorphic functions on time scales and to start the investigation of existence and uniqueness of almost automorphic solutions of dynamic equations. More precisely, we study the nonautonomous dynamic equations on time scales given by

$$x^\Delta(t) = A(t)x(t) + f(t), \quad t \in \mathbb{T}, \quad (1.1)$$

where $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n})$ and $f \in C_{rd}(\mathbb{T}, \mathbb{R}^n)$.

We prove the existence of an almost automorphic solution of (1.1), assuming the associated homogeneous equation of (1.1) admits an exponential dichotomy and \mathbb{T} is an invariant under translations time scale, concept that we introduce here for the first time. In passing, we show that in these time scales the graininess function has the remarkable property of being automatically almost automorphic. Also, we suppose $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n})$ is almost automorphic and nonsingular matrix function, the sets

$$\{A^{-1}(t)\}_{t \in \mathbb{T}} \quad \text{and} \quad \{(I + \mu(t)A(t))^{-1}\}_{t \in \mathbb{T}} \quad (1.2)$$

are bounded and $f \in C_{rd}(\mathbb{T}, \mathbb{R}^n)$ is almost automorphic function.

After that, we consider the semilinear dynamic equation on time scales given by

$$x^\Delta(t) = A(t)x(t) + f(t, x(t)), \quad t \in \mathbb{T}, \quad (1.3)$$

where $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n})$, $f \in C_{rd}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ and \mathbb{T} is an invariant under translations time scale.

We also obtain the existence and uniqueness of an almost automorphic solution of (1.3), we assume the associated homogeneous equation of (1.3) admits an exponential dichotomy and $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n})$ is almost automorphic and nonsingular matrix function, the sets in (1.2) are bounded and $f \in C_{rd}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ is almost automorphic function with respect to first variable and satisfies the global Lipschitz condition with respect to the second variable.

Moreover, we present some applications of our results for new and interesting invariant under translations time scales. Finally, we present interesting models in which our results can be applied.

The present paper is organized as follows. The second section is devoted to present the preliminary results concerning the theory of time scales. In the third section, we prove some properties for almost automorphic functions on time scales and present some examples. The fourth section is devoted to present some basic concepts and main results concerning product integration on time scales. The fifth section brings a result which ensures the existence of almost automorphic solutions for linear dynamic equations on time scales. In the sixth section, we prove an existence and uniqueness of almost automorphic solutions for semilinear dynamic equations on time scales. Finally, the last section is devoted to present some interesting examples and applications of our main results.

2. Preliminaries

In this section, we present some basic concepts and results concerning time scales which will be essential to prove our main results. For more details, the reader may want to consult [11] and [12].

Let \mathbb{T} be a time scale, that is, closed and nonempty subset of \mathbb{R} . For every $t \in \mathbb{T}$, we define the forward and backward jump operators $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$, respectively, as follows:

$$\sigma(t) = \inf\{s \in \mathbb{T}, s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T}, s < t\}.$$

In this definition, we consider $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

If $\sigma(t) > t$, we say that t is *right-scattered*. Otherwise, t is called *right-dense*. Analogously, if $\rho(t) < t$, then t is called *left-scattered* whereas if $\rho(t) = t$, then t is *left-dense*.

We also define the *graininess function* $\mu : \mathbb{T} \rightarrow \mathbb{R}^+$ and the *backward graininess function* $\nu : \mathbb{T} \rightarrow \mathbb{R}^+$, respectively, by

$$\mu(t) = \sigma(t) - t \quad \text{and} \quad \nu(t) = t - \rho(t).$$

Definition 2.1. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *rd-continuous* if it is regulated on \mathbb{T} and continuous at right-dense points of \mathbb{T} . If the function $f : \mathbb{T} \rightarrow \mathbb{R}$ is continuous at each right-dense point and each left-dense point, then the function f is said to be *continuous* on \mathbb{T} . We denote the class of all rd-continuous functions $f : \mathbb{T} \rightarrow \mathbb{R}$ by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

Given a pair of numbers $a, b \in \mathbb{T}$, the symbol $[a, b]_{\mathbb{T}}$ will be used to denote a closed interval in \mathbb{T} , that is, $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T}; a \leq t \leq b\}$. On the other hand, $[a, b]$ is the usual closed interval on the real line, that is, $[a, b] = \{t \in \mathbb{R}; a \leq t \leq b\}$.

We define the set \mathbb{T}^k which is derived from \mathbb{T} as follows: If \mathbb{T} has a left-scattered maximum m , then $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^k = \mathbb{T}$.

Definition 2.2. For $y : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the *delta-derivative* of y to be the number (if it exists) with the following property: given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|y(\sigma(t)) - y(t) - y^\Delta(t)[\sigma(t) - s]| < \varepsilon |\sigma(t) - s|,$$

for all $s \in U$.

Similarly, we can define the nabla-derivative of the function $y : \mathbb{T} \rightarrow \mathbb{R}$. For details, see [11] and [12].

Definition 2.3. A *partition* of $[a, b]_{\mathbb{T}}$ is a finite sequence of points

$$\{t_0, t_1, \dots, t_m\} \subset [a, b]_{\mathbb{T}}, \quad a = t_0 < t_1 < \dots < t_m = b.$$

Given such a partition, we put $\Delta t_i = t_i - t_{i-1}$. A tagged partition consists of a partition and a sequence of tags $\{\xi_1, \dots, \xi_m\}$ such that $\xi_i \in [t_{i-1}, t_i)$ for every $i \in \{1, \dots, m\}$. The set of all tagged partitions of $[a, b]_{\mathbb{T}}$ will be denoted by the symbol $D(a, b)$.

If $\delta > 0$, then $D_\delta(a, b)$ denotes the set of all tagged partitions of $[a, b]_{\mathbb{T}}$ such that for every $i \in \{1, \dots, m\}$, either $\Delta t_i \leq \delta$, or $\Delta t_i > \delta$ and $\sigma(t_{i-1}) = t_i$. Note that in the last case, the only way to choose a tag in $[t_{i-1}, t_i)$ is to take $\xi_i = t_{i-1}$.

In the sequel, we present the definition of Riemann Δ -integrals. See [11] and [12], for instance.

Definition 2.4. We say that f is *Riemann Δ -integrable* on $[a, b]_{\mathbb{T}}$, if there exists a number I with the following property: for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| \sum_i f(\xi_i)(t_i - t_{i-1}) - I \right| < \varepsilon,$$

for every $P \in D_\delta(a, b)$ independently of $\xi_i \in [t_{i-1}, t_i)_{\mathbb{T}}$ for $1 \leq i \leq n$. It is clear that such a number I is unique and is the Riemann Δ -integral of f from a to b .

Similarly, we can define the Riemann ∇ -integrable functions on $[a, b]_{\mathbb{T}}$. See [11] and [12], for instance.

In what follows, we present a concept of regressive functions.

Definition 2.5. We say that a function $p : \mathbb{T} \rightarrow \mathbb{R}$ is *regressive* provided

$$1 + \mu(t)p(t) \neq 0, \quad \text{for all } t \in \mathbb{T}^k$$

holds. The set of all regressive and rd-continuous functions will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Suppose that $p, q \in \mathcal{R}$, then we define $p \oplus q$ and $\ominus p$ as follows:

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t), \quad \text{for all } t \in \mathbb{T}^k$$

and

$$(\ominus p)(t) := \frac{-p(t)}{1 + \mu(t)p(t)}, \quad \text{for all } t \in \mathbb{T}^k.$$

It is clear that (\mathcal{R}, \oplus) is an Abelian group. (See, for instance, [11].) In the sequel, we define the *generalized exponential function* $e_p(t, s)$.

Definition 2.6. If $p \in \mathcal{R}$, then we define the *generalized exponential function* by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta\tau\right) \quad \text{for } s, t \in \mathbb{T},$$

where the cylinder transformation $\xi_h : \mathbb{C}_h \rightarrow \mathbb{Z}_h$ is given by

$$\xi_h(z) = \frac{1}{h} \log(1 + zh),$$

where \log is the principal logarithm function. For $h = 0$, we define $\xi_0(z) = z$ for all $z \in \mathbb{C}$.

In what follows, we present some definitions about matrix-valued functions on time scales.

Definition 2.7. Let A be an $m \times n$ matrix-valued function on \mathbb{T} . We say that A is *rd-continuous* on \mathbb{T} if each entry of A is rd-continuous on \mathbb{T} . We denote the class of all rd-continuous $m \times n$ matrix-valued functions on \mathbb{T} by $\mathcal{C}_{rd} = \mathcal{C}_{rd}(\mathbb{T}) = \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}^{m \times n})$.

We say that A is *delta-differentiable* at \mathbb{T} if each entry of A is delta-differentiable on \mathbb{T} . And in this case, we have

$$A^\sigma(t) = A(t) + \mu(t)A^\Delta(t).$$

Definition 2.8. An $m \times n$ matrix-valued function A on a time scale \mathbb{T} is called *regressive* (with respect to \mathbb{T}) provided

$$I + \mu(t)A(t) \text{ is invertible} \quad \text{for all } t \in \mathbb{T}^k,$$

and the class of all such regressive rd-continuous functions is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}^{m \times n})$.

Assume that A and B are regressive $n \times n$ matrix-valued functions on \mathbb{T} . Then, we define $A \oplus B$ by

$$(A \oplus B)(t) = A(t) + B(t) + \mu(t)A(t)B(t), \quad \forall t \in \mathbb{T}^k,$$

and we define $\ominus A$ by

$$(\ominus A)(t) = -[I + \mu(t)A(t)]^{-1}A(t), \quad \forall t \in \mathbb{T}^k.$$

It is clear that $(\mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n}), \oplus)$ is a group. For more details, see [11].

We proceed giving the definition of matrix exponential function found in [11].

Definition 2.9 (*Matrix exponential function*). Let $t_0 \in \mathbb{T}$ and assume that $A \in \mathcal{R}$ is an $n \times n$ matrix-valued function. The unique matrix-valued solution of the IVP

$$Y^\Delta(t) = A(t)Y(t), \quad Y(t_0) = I,$$

where I denotes as usual the $n \times n$ -identity matrix, is called the *matrix exponential function* at t_0 and it is denoted by $e_A(\cdot, t_0)$.

In the sequel, we enunciate a result which describes the properties of matrix exponential function. It can be found in [11, Theorem 5.21].

Theorem 2.10. *If $A, B \in \mathcal{R}$ are matrix-valued functions on \mathbb{T} , then*

- (i) $e_0(t, s) \equiv I$ and $e_A(t, t) \equiv I$;
- (ii) $e_A(\sigma(t), s) = (I + \mu(t)A(t))e_A(t, s)$;
- (iii) $e_A^{-1}(t, s) = e_{\ominus A^*}^*(t, s)$;
- (iv) $e_A(t, s) = e_A^{-1}(s, t) = e_{\ominus A^*}^*(s, t)$;
- (v) $e_A(t, s)e_A(s, r) = e_A(t, r)$;
- (vi) $e_A(t, s)e_B(t, s) = e_{A \oplus B}(t, s)$ if $e_A(t, s)$ and $B(t)$ commute.

Using these notions, one can obtain the following result which is a variation of constants formula which can be found in [11, Theorem 5.24].

Theorem 2.11 (*Variation of constants formula*). Let $A \in \mathcal{R}$ be an $n \times n$ matrix-valued function on \mathbb{T} and suppose that $f : \mathbb{T} \rightarrow \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then the initial value problem

$$\begin{cases} y^\Delta(t) = A(t)y(t) + f(t), \\ y(t_0) = y_0 \end{cases} \quad (2.1)$$

has a unique solution $y : \mathbb{T} \rightarrow \mathbb{R}^n$. Moreover, this solution is given by

$$y(t) = e_A(t, t_0)y_0 + \int_{t_0}^t e_A(t, \sigma(\tau))f(\tau)\Delta\tau.$$

Definition 2.12. Let $A(t)$ be an $n \times n$ rd-continuous matrix-valued function on \mathbb{T} . We say that the linear system

$$x^\Delta(t) = A(t)x(t) \quad (2.2)$$

has an *exponential dichotomy* on \mathbb{T} if there exist positive constants K and γ , projection P , which commutes with $X(t)$, $t \in \mathbb{T}$, and fundamental solution matrix $X(t)$ of (2.2) satisfying

$$\begin{aligned} |X(t)PX^{-1}(s)| &\leq Ke_{\ominus\gamma}(t, s), \quad s, t \in \mathbb{T}, t \geq s, \\ |X(t)(I - P)X^{-1}(s)| &\leq Ke_{\ominus\gamma}(s, t), \quad s, t \in \mathbb{T}, t \leq s. \end{aligned}$$

The following result will be essential to our purposes. For a proof of this result, see [11, Theorem 2.39].

Theorem 2.13. If $p \in \mathcal{R}$ and $a, b, c \in \mathbb{T}$, then

$$[e_p(c, \cdot)]^\Delta = -p[e_p(c, \cdot)]^\sigma$$

and

$$\int_a^b p(t)e_p(c, \sigma(t))\Delta t = e_p(c, a) - e_p(c, b).$$

The following result shows that $e_{\ominus\alpha}(t, s)$, for $\alpha > 0$, $t > s$, is a bounded function. The proof is inspired by [38, Lemma 5.1].

Theorem 2.14. If $\alpha > 0$, then $e_{\ominus\alpha}(t, s) \leq 1$ for $t, s \in \mathbb{T}$ such that $t > s$.

Proof. First, we suppose $\mu(t) = 0$, then

$$e_{\ominus\alpha}(t, s) = \exp\left(\int_s^t \xi_0(\ominus\alpha)\Delta\tau\right),$$

for every $s, t \in \mathbb{T}$. Then,

$$e_{\ominus\alpha}(t, s) = \exp\left(\int_s^t \ominus\alpha\Delta\tau\right) = \exp\left(\int_s^t \frac{-\alpha}{1 + \mu(\tau)\alpha}\Delta\tau\right) = \exp(-\alpha(t - s)),$$

using the fact that $\mu(\tau) = 0$. Since $t > s$, we obtain

$$e_{\ominus\alpha}(t, s) = \exp(-\alpha(t - s)) \leq 1.$$

Now, let us consider $\mu(t) > 0$, then

$$1 + \mu(t) \ominus \alpha = 1 + \mu(t) \frac{-\alpha}{1 + \mu(t)\alpha} = \frac{1 + \mu(t)\alpha - \mu(t)\alpha}{1 + \mu(t)\alpha} = \frac{1}{1 + \mu(t)\alpha} < 1.$$

Thus, $\ominus\alpha \in \mathcal{R}$ and it is easy to see that

$$\log(1 + \mu(t) \ominus \alpha) \in \mathbb{R}$$

for all $t \in \mathbb{T}$. Then, it follows

$$\xi_{\mu(t)}(\ominus\alpha) = \frac{\log(1 + \mu(t) \ominus \alpha)}{\mu(t)} < 0,$$

which implies that

$$e_{\ominus\alpha}(t, s) = \exp\left(\int_s^t \xi_{\mu(t)}(\ominus\alpha)\right) < 1,$$

for every $t, s \in \mathbb{T}$ such that $t > s$. \square

The next result describes the solution of (2.1). It can be found in [37, Lemma 2.13].

Theorem 2.15. *If the linear system (2.2) admits exponential dichotomy, then the system (2.1) has a bounded solution $x(t)$ as follows:*

$$x(t) = \int_{-\infty}^t X(t) P X^{-1}(\sigma(s)) f(s) \Delta s - \int_t^{+\infty} X(t) (I - P) X^{-1}(\sigma(s)) f(s) \Delta s,$$

where $X(t)$ is the fundamental solution matrix of (2.2).

3. Almost automorphic functions on time scales

In this section, we introduce almost automorphic functions on time scales and present their properties.

We start by introducing a definition of an invariant under translations time scale.

Definition 3.1. A time scale \mathbb{T} is called *invariant under translations* if

$$\Pi := \{\tau \in \mathbb{R}: t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}. \quad (3.1)$$

We say that the graininess function $\mu: \mathbb{T} \rightarrow \mathbb{R}_+$ is an *almost automorphic function* if for every sequence (α'_n) on Π , there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n \rightarrow \infty} \mu(t + \alpha_n) = \bar{\mu}(t), \quad (3.2)$$

for every $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} \bar{\mu}(t - \alpha_n) = \mu(t), \quad (3.3)$$

for every $t \in \mathbb{T}$.

Combining Theorems 1.9, 1.10 and 1.11 from [21], we obtain the following characterization of almost periodic functions $f : \mathbb{R} \rightarrow \mathbb{R}$.

Theorem 3.2. *The function $f : \mathbb{R} \rightarrow \mathbb{R}$ is almost periodic, if and only if, from any sequence of the form $\{f(x + \alpha_n)\}$, where (α_n) is a sequence of real numbers, one can extract a subsequence converging uniformly on the real line.*

A carefully examination of the proofs of Theorems 1.9, 1.10 and 1.11 from [21] reveals that the result above remains true for a general time scale. More precisely, we obtain the next result.

Theorem 3.3. *Let \mathbb{T} be an invariant under translations time scale, then the function $f : \mathbb{T} \rightarrow \mathbb{R}$ is almost periodic, if and only if, from any sequence of the form $\{f(x + \alpha_n)\}$, where (α_n) is a sequence on Π , one can extract a subsequence converging uniformly on \mathbb{T} .*

Therefore, using this fact and the definition of invariant under translations time scales, we get the following result.

Theorem 3.4. *If \mathbb{T} is an invariant under translations time scale, then the graininess function $\mu : \mathbb{T} \rightarrow \mathbb{R}_+$ is an almost periodic function.*

Proof. If the time scale \mathbb{T} is invariant under translation, then Eq. (3.1) is satisfied. Then, let us consider two cases: if all the points in \mathbb{T} are right-dense (or/and left-dense) and otherwise.

Let us consider that all the points in \mathbb{T} are right-dense. (Notice that the cases in which all points in \mathbb{T} are left-dense, or even are right-dense and left-dense at the same time, follow similarly. Thus, we will prove only this case.) From this fact and since \mathbb{T} is invariant under translation, we obtain that $\mathbb{T} = \mathbb{R}$, because the condition (3.1) must be satisfied. Therefore, in this case, it follows that $\mu(t) = 0$ for every $t \in \mathbb{T}$ and the almost periodicity of the graininess function follows immediately.

Now, let us suppose that \mathbb{T} has at least one point which is not right-dense (or left-dense), then in this case, it makes sense to consider $\min\{|\tau| : \tau \in \Pi\}$, which is clearly finite, since $\tau \in \mathbb{R}$. Thus, denote $K := \min\{|\tau| : \tau \in \Pi\}$.

Then, by the definition of forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$, we have

$$\sigma(t) \leq t + K,$$

which implies that $\mu(t) \leq K$, for every $t \in \mathbb{T}$.

Given a sequence $(\alpha_n) \in \Pi$, define $\mu_n(t) := \mu(t + \alpha_n)$. Obviously, by the properties of an invariant under translations time scale, we have $\mu_n : \mathbb{T} \rightarrow \mathbb{R}_+$. Therefore, since the function μ_n takes value on \mathbb{R}_+ and is bounded ($0 \leq \mu_n(t) \leq K$), we obtain by Bolzano–Weierstrass Theorem that μ_n possesses a subsequence which converges uniformly.

Thus, by Theorem 3.3 and using the fact that $\{\mu(t + \alpha_n)\}$ possesses a subsequence which converges uniformly, we obtain that μ is an almost periodic function. \square

Using the fact that every almost periodic function is almost automorphic, we obtain as an immediate consequence of the previous theorem the following result.

Corollary 3.5. *If \mathbb{T} is an invariant under translations time scales, then the graininess function $\mu : \mathbb{T} \rightarrow \mathbb{R}_+$ is almost automorphic.*

Remark 3.6. We point out that in the paper [37], the authors use the same definition of an invariant under translation time scale presented here to define an almost periodic time scale.

However, by Corollary 3.5, one can see that this concept is more general, since it can be also applied to study almost automorphic functions. Therefore, we have rewritten the definition presented in [37] and called the time scale which satisfies the property (3.1) as *invariant under translations*.

If $\tau_1, \tau_2 \in \Pi$, then $\tau_1 \pm \tau_2 \in \Pi$ and \mathbb{T} is an invariant under translations time scale, then $\inf \mathbb{T} = -\infty$ and $\sup \mathbb{T} = +\infty$.

In what follows, we give some interesting examples of invariant under translations time scales \mathbb{T} .

Example 3.7. The time scales $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{R}$ are clearly invariant under translations.

Example 3.8. Notice that $\mathbb{T} = h\mathbb{Z}$, for $h \in \mathbb{Z}$ and $\mathbb{T} = \frac{1}{n}\mathbb{Z}$, $n \in \mathbb{N}_0 = \mathbb{N} \setminus \{0\}$ are invariant under translations time scales.

Example 3.9. Consider the time scale

$$\mathbb{P}_{a,b} = \bigcup_{k=-\infty}^{\infty} [k(a+b), k(a+b)+a]$$

then

$$\sigma(t) = \begin{cases} t, & \text{if } t \in \bigcup_{k=-\infty}^{\infty} [k(a+b), k(a+b)+a), \\ t+b, & \text{if } t \in \bigcup_{k=-\infty}^{\infty} \{k(a+b)+a\} \end{cases}$$

and

$$\mu(t) = \begin{cases} 0, & \text{if } t \in \bigcup_{k=-\infty}^{\infty} [k(a+b), k(a+b)+a), \\ b, & \text{if } t \in \bigcup_{k=-\infty}^{\infty} \{k(a+b)+a\}. \end{cases}$$

By the definition, it follows that $\mathbb{P}_{a,b}$ is an invariant under translation time scale.

Considering $a = 1$ and $b = 1$ in Example 3.9, we obtain that $\mathbb{P}_{1,1}$ describes the population of certain species which its life span is one unit of time. In other words, just before the species dies out, eggs are laid which are hatched one unit of time later. For this specific case, see [19].

The following two examples bring more complex time scales. They might describe a population of certain species which its life span behaves the same way as a cosine and sin function, respectively.

Example 3.10. Let $0 < a < \frac{\pi}{2}$ and consider the time scale

$$\mathbb{P}_{a, \cos a} = \bigcup_{k=-\infty}^{\infty} [k(a + \cos a), k(a + \cos a) + a]$$

then

$$\sigma(t) = \begin{cases} t, & \text{if } t \in \bigcup_{k=-\infty}^{\infty} [k(a + \cos a), k(a + \cos a) + a), \\ t + \cos t, & \text{if } t \in \bigcup_{k=-\infty}^{\infty} \{k(a + \cos a) + a\} \end{cases}$$

and

$$\mu(t) = \begin{cases} 0, & \text{if } t \in \bigcup_{k=-\infty}^{\infty} [k(a + \cos a), k(a + \cos a) + a), \\ \cos t, & \text{if } t \in \bigcup_{k=-\infty}^{\infty} \{k(a + \cos a) + a\}. \end{cases}$$

It is clear that $\mathbb{P}_{a, \cos a}$ is an invariant under translations time scale.

Example 3.11. Let $\frac{\pi}{2} < a < \pi$ and consider the time scale

$$\mathbb{P}_{a, \sin a} = \bigcup_{k=-\infty}^{\infty} [k(a + \sin a), k(a + \sin a) + a]$$

then

$$\sigma(t) = \begin{cases} t, & \text{if } t \in \bigcup_{k=-\infty}^{\infty} [k(a + \sin a), k(a + \sin a) + a), \\ t + \sin t, & \text{if } t \in \bigcup_{k=-\infty}^{\infty} \{k(a + \sin a) + a\} \end{cases}$$

and

$$\mu(t) = \begin{cases} 0, & \text{if } t \in \bigcup_{k=-\infty}^{\infty} [k(a + \sin a), k(a + \sin a) + a), \\ \sin t, & \text{if } t \in \bigcup_{k=-\infty}^{\infty} \{k(a + \sin a) + a\}. \end{cases}$$

Clearly, $\mathbb{P}_{a, \sin a}$ is an invariant under translations time scale.

In the sequel, we present some time scales which are not invariant under translations.

Example 3.12. Clearly, $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$, $q > 1$, is not invariant under translations, since \mathbb{T} does not satisfy the condition (3.1). Note that $\mu(t) = (q - 1)t$.

Example 3.13. Every compact interval $\mathbb{T} = [a, b]$, $a, b \in \mathbb{R}$, is not invariant under translations.

Example 3.14. The time scales $\mathbb{T} = \mathbb{N}_0^2$ and $\mathbb{T} = 2^{\mathbb{N}}$ are not invariant under translations. Here $\mu(t) = 2\sqrt{t} + 1$ and $\mu(t) = t$ respectively.

Now, we introduce the definition of an *almost automorphic function* on time scales.

Definition 3.15. Let X be a (real or complex) Banach space and \mathbb{T} be an invariant under translation time scale. Then, an rd-continuous function $f : \mathbb{T} \rightarrow X$ is called *almost automorphic* on \mathbb{T} if for every sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n \rightarrow \infty} f(t + \alpha_n) = \bar{f}(t)$$

is well defined for each $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} \bar{f}(t - \alpha_n) = f(t),$$

for every $t \in \mathbb{T}$.

We denote the space of all almost automorphic function on time scales $f : \mathbb{T} \rightarrow X$ by $AA_{\mathbb{T}}(X)$.

In what follows, we present some properties concerning almost automorphic function on time scale \mathbb{T} . The proof is inspired by Theorems 2.1.3 and 2.1.4, from [42].

Theorem 3.16. Let \mathbb{T} be an invariant under translations time scale and suppose the rd-continuous functions $f, g : \mathbb{T} \rightarrow X$ are almost automorphic on time scales. Then the following assertions hold.

- (i) $f + g$ is almost automorphic function on time scales;
- (ii) cf is almost automorphic function on time scales for every scalar c ;
- (iii) For each $l \in \mathbb{T}$, the function $f_l : \mathbb{T} \rightarrow X$ defined by $f_l(t) := f(l + t)$ is almost automorphic on time scales;
- (iv) The function $\hat{f} : \mathbb{T} \rightarrow X$ defined by $\hat{f}(t) := f(-t)$ is almost automorphic on time scales;
- (v) $\sup_{t \in \mathbb{T}} \|f(t)\| < \infty$, that is, f is a bounded function;
- (vi) $\sup_{t \in \mathbb{T}} \|\bar{f}(t)\| \leq \sup_{t \in \mathbb{T}} \|f(t)\|$, where

$$\lim_{n \rightarrow \infty} f(t + \alpha_n) = \bar{f}(t) \quad \text{and} \quad \lim_{n \rightarrow \infty} \bar{f}(t - \alpha_n) = f(t).$$

Proof. Let $f, g : \mathbb{T} \rightarrow X$ be almost automorphic functions on time scales. Then, for every sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n \rightarrow \infty} f(t + \alpha_n) = \bar{f}(t) \quad \text{and} \quad \lim_{n \rightarrow \infty} g(t + \alpha_n) = \bar{g}(t)$$

are well defined for each $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} \bar{f}(t - \alpha_n) = f(t) \quad \text{and} \quad \lim_{n \rightarrow \infty} \bar{g}(t - \alpha_n) = g(t),$$

for every $t \in \mathbb{T}$. Thus, we obtain

$$\lim_{n \rightarrow \infty} (f + g)(t + \alpha_n) := \bar{f}(t) + \bar{g}(t)$$

is well defined for each $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} (\bar{f} + \bar{g})(t - \alpha_n) = f(t) + g(t),$$

for every $t \in \mathbb{T}$. Thus, item (i) follows.

Since f is almost automorphic function on time scales, then for every sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n \rightarrow \infty} (cf)(t + \alpha_n) = \lim_{n \rightarrow \infty} cf(t + \alpha_n) = c\bar{f}(t) = (c\bar{f})(t)$$

is well defined for each $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} (c\bar{f})(t - \alpha_n) = \lim_{n \rightarrow \infty} c\bar{f}(t - \alpha_n) = cf(t) = (cf)(t),$$

for every $t \in \mathbb{T}$, which proves item (ii).

The proofs of items (iii) and (iv) follow using similar argument as before. Thus, we omit them here.

Let us prove item (v). Let $t_0 \in \mathbb{T}$ and suppose that $\sup_{k \in \mathbb{T}} \|f(k)\| = \infty$, then there exists a sequence $(\alpha'_n) \subset \Pi$ such that

$$\lim_{n \rightarrow \infty} \|f(t_0 + \alpha'_n)\| = \infty.$$

Since f is almost automorphic, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n \rightarrow \infty} f(t_0 + \alpha_n) = \bar{f}(t_0),$$

and using the continuity of norm function, we get

$$\lim_{n \rightarrow \infty} \|f(t_0 + \alpha_n)\| = \|\bar{f}(t_0)\| < \infty,$$

which contradicts the fact that $\lim_{n \rightarrow \infty} \|f(t_0 + \alpha'_n)\| = \infty$.

Finally, let us prove item (vi). Let (α'_n) be a sequence on Π , then there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\|\bar{f}(t)\| = \left\| \lim_{n \rightarrow \infty} f(t + \alpha_n) \right\| = \lim_{n \rightarrow \infty} \|f(t + \alpha_n)\| \leq \sup_{t \in \mathbb{T}} \|f(t)\|,$$

which implies that

$$\sup_{t \in \mathbb{T}} \|\bar{f}(t)\| \leq \sup_{t \in \mathbb{T}} \|f(t)\|. \quad (3.4)$$

On the other hand, we have

$$\|f(t)\| = \left\| \lim_{n \rightarrow \infty} \bar{f}(t + \alpha_n) \right\| \leq \sup_{t \in \mathbb{T}} \|\bar{f}(t)\|,$$

then

$$\sup_{t \in \mathbb{T}} \|f(t)\| \leq \sup_{t \in \mathbb{T}} \|\tilde{f}(t)\|. \quad (3.5)$$

Combining (3.4) and (3.5), we obtain

$$\sup_{t \in \mathbb{T}} \|f(t)\| = \sup_{t \in \mathbb{T}} \|\tilde{f}(t)\|,$$

and the result follows as well. \square

The next result generalizes item (ii) from Theorem 3.16. The proof is inspired by Theorem 2.7, from [3].

Theorem 3.17. *Let \mathbb{T} be an invariant under translations time scale and the functions $f, u : \mathbb{T} \rightarrow X$ be almost automorphic on time scales, then the function $uf : \mathbb{T} \rightarrow X$ defined by $(uf)(t) = u(t)f(t)$ is almost automorphic on time scales.*

Proof. Let (α'_n) be a sequence on Π , then there exists a subsequence $(\alpha''_n) \subset (\alpha'_n)$ such that $\lim_{n \rightarrow \infty} u(t + \alpha''_n) = \bar{u}(t)$ is well defined for each $t \in \mathbb{T}$ and $\lim_{n \rightarrow \infty} \bar{u}(t - \alpha''_n) = u(t)$ for each $t \in \mathbb{T}$. Since f is almost automorphic, there exists a subsequence $(\alpha_n) \subset (\alpha''_n)$ such that $\lim_{n \rightarrow \infty} f(t + \alpha_n) = \tilde{f}(t)$ is well defined for each $t \in \mathbb{T}$ and $\lim_{n \rightarrow \infty} \tilde{f}(t - \alpha_n) = f(t)$ for each $t \in \mathbb{T}$. Thus,

$$\begin{aligned} \|u(t + \alpha_n)f(t + \alpha_n) - \bar{u}(t)\tilde{f}(t)\| &\leq \|u(t + \alpha_n)(f(t + \alpha_n) - \tilde{f}(t))\| \\ &\quad + \|(u(t + \alpha_n) - \bar{u}(t))\tilde{f}(t)\| \\ &< \varepsilon, \end{aligned}$$

for n sufficiently large. Therefore,

$$\lim_{n \rightarrow \infty} u(t + \alpha_n)f(t + \alpha_n) = \bar{u}(t)\tilde{f}(t),$$

for every $t \in \mathbb{T}$. Analogously, we can prove that $\lim_{n \rightarrow \infty} \bar{u}(t - \alpha_n)\tilde{f}(t - \alpha_n) = u(t)f(t)$. Thus, we obtain that uf is an almost automorphic function on time scales. \square

In the sequel, we present a result which ensures that $AA_{\mathbb{T}}(X)$ is a Banach space with the norm described in item (v) from Theorem 3.16. The proof is inspired by [42, Theorem 2.1.10].

Theorem 3.18. *Let \mathbb{T} be an invariant under translations time scale and (f_n) be a sequence of almost automorphic functions such that $\lim_{n \rightarrow \infty} f_n(t) = f(t)$ converges uniformly for each $t \in \mathbb{T}$. Then, f is an almost automorphic function.*

Proof. Let (α'_n) be a sequence on Π . Since $f_1 \in AA_{\mathbb{T}}(X)$, then there exists a subsequence $(\alpha''_n) \subset (\alpha'_n)$ such that

$$\lim_{n \rightarrow \infty} f_1(t + \alpha_n^1) := \bar{f}_1(t)$$

is well defined for every $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} \bar{f}_1(t - \alpha_n^1) = f_1(t),$$

for every $t \in \mathbb{T}$. Since $f_2 \in AA_{\mathbb{T}}(X)$, then there exists a subsequence $(\alpha_n^2) \subset (\alpha_n^1)$ such that

$$\lim_{n \rightarrow \infty} f_2(t + \alpha_n^2) = \bar{f}_2(t)$$

is well defined for every $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} \bar{f}_2(t - \alpha_n^2) = f_2(t),$$

for every $t \in \mathbb{T}$. Thus, by the diagonal procedure, we can construct a subsequence $(\alpha_n) \subset (\alpha_n')$ such that

$$\lim_{n \rightarrow \infty} f_i(t + \alpha_n) = \bar{f}_i(t), \quad (3.6)$$

for each $t \in \mathbb{T}$ and for all $i = 1, 2, 3, \dots$. Notice that

$$\begin{aligned} \|\bar{f}_i(t) - \bar{f}_j(t)\| &\leq \|\bar{f}_i(t) - f_i(t + \alpha_n)\| + \|f_i(t + \alpha_n) - f_j(t + \alpha_n)\| \\ &\quad + \|f_j(t + \alpha_n) - \bar{f}_j(t)\|. \end{aligned} \quad (3.7)$$

Let $\varepsilon > 0$, then by the uniform convergence of (f_n) , we can find $N \in \mathbb{N}$ sufficiently large such that for all $i, j > N$, we obtain

$$\|f_i(t + \alpha_n) - f_j(t + \alpha_n)\| < \varepsilon, \quad (3.8)$$

for all $t \in \mathbb{T}$ and all $n = 1, 2, \dots$.

Therefore, taking i, j sufficiently large in (3.7) and using (3.8) and the limit (3.6), we obtain that $(\bar{f}_i(t))$ is a Cauchy sequence. Since X is a Banach space, then $(\bar{f}_i(t))$ is a sequence which converges pointwisely on X . Let $\bar{f}(t)$ be the limit of $(\bar{f}_i(t))$, then for each $i = 1, 2, 3, \dots$, we have

$$\begin{aligned} \|f(t + \alpha_n) - \bar{f}(t)\| &\leq \|f(t + \alpha_n) - f_i(t + \alpha_n)\| + \|f_i(t + \alpha_n) - \bar{f}_i(t)\| \\ &\quad + \|\bar{f}_i(t) - \bar{f}(t)\|. \end{aligned} \quad (3.9)$$

Then, for i sufficiently large, by (3.9) and using the almost automorphicity of f_i and the convergence of the functions f_i and \bar{f}_i , we obtain

$$\lim_{n \rightarrow \infty} f(t + \alpha_n) = \bar{f}(t),$$

for each $t \in \mathbb{T}$. Analogously, one can prove that

$$\lim_{n \rightarrow \infty} \bar{f}(t - \alpha_n) = f(t),$$

for every $t \in \mathbb{T}$ and we get the desired result. \square

In what follows, we present a result which brings a property concerning composition of almost automorphic function on time scales and a continuous function. The proof is inspired by [3, Theorem 2.5].

Theorem 3.19. *Let \mathbb{T} be an invariant under translations time scale and let X, Y be Banach spaces. Suppose $f : \mathbb{T} \rightarrow X$ is an almost automorphic function on time scales and $\phi : X \rightarrow Y$ is a continuous function, then the composite function $\phi \circ f : \mathbb{T} \rightarrow Y$ is an almost automorphic function on time scales.*

Proof. Since $f \in AA_{\mathbb{T}}(X)$, for every sequence (α'_n) on Π , there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that $\lim_{n \rightarrow \infty} f(t + \alpha_n) = \bar{f}(t)$ is well defined for every $t \in \mathbb{T}$ and $\lim_{n \rightarrow \infty} \bar{f}(t - \alpha_n) = f(t)$ for each $t \in \mathbb{T}$.

By the continuity of function ϕ , it follows that

$$\lim_{n \rightarrow \infty} \phi(f(t + \alpha_n)) = \phi\left(\lim_{n \rightarrow \infty} f(t + \alpha_n)\right) = (\phi \circ \bar{f})(t).$$

Similarly, we have

$$\lim_{n \rightarrow \infty} \phi(\bar{f}(t - \alpha_n)) = \phi\left(\lim_{n \rightarrow \infty} \bar{f}(t - \alpha_n)\right) = (\phi \circ f)(t)$$

for each $t \in \mathbb{T}$. Thus, $\phi \circ f \in AA_{\mathbb{T}}(Y)$. \square

Now, we present the definition of an almost automorphic function on time scales depending on one parameter. This definition is useful for applications to nonlinear dynamic equations.

Definition 3.20. Let X be a (real or complex) Banach space and \mathbb{T} be an invariant under translations time scale. Then, an rd-continuous function $f : \mathbb{T} \times X \rightarrow X$ is called *almost automorphic on $t \in \mathbb{T}$* for each $x \in X$, if for every sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n \rightarrow \infty} f(t + \alpha_n, x) = \bar{f}(t, x) \tag{3.10}$$

is well defined for each $t \in \mathbb{T}$, $x \in X$ and

$$\lim_{n \rightarrow \infty} \bar{f}(t - \alpha_n, x) = f(t, x), \tag{3.11}$$

for every $t \in \mathbb{T}$ and $x \in X$.

In the sequel, we present a result concerning the properties of almost automorphic functions on time scales with respect to the first variable. We omit the proof since it is similar to the proof of [Theorem 3.16](#).

Theorem 3.21. *Let \mathbb{T} be an invariant under translations time scale and $f, g : \mathbb{T} \times X \rightarrow X$ be almost automorphic functions on time scales in t for each x in X . Then the following assertions hold.*

- (i) $f + g$ is almost automorphic function on time scales in t for each x in X .
- (ii) cf is almost automorphic function on time scales in t for each x in X , where c is an arbitrary scalar.
- (iii) $\sup_{t \in \mathbb{T}} \|f(t, x)\| = M_x < \infty$, for each x in X .
- (iv) $\sup_{t \in \mathbb{T}} \|\bar{f}(t, x)\| = N_x < \infty$, for each x in X , where \bar{f} is the function in [Definition 3.20](#).

Now, we present a result which will be essential to prove the following one. The proof is inspired by Theorem 2.2.5 from [\[42\]](#).

Theorem 3.22. *Let \mathbb{T} be an invariant under translations time scale and $f : \mathbb{T} \times X \rightarrow X$ be an almost automorphic function on time scales for each $x \in X$ and if f is Lipschitzian in x uniformly in t , then \bar{f} given by [\(3.10\)](#) and [\(3.11\)](#) satisfies the same Lipschitz condition in x uniformly in t .*

Proof. Let $L > 0$ be a Lipschitz constant for the function f , that is, the following inequality

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|$$

holds for every $x, y \in X$ uniformly in $t \in \mathbb{T}$.

Let $t \in \mathbb{T}$ be arbitrary and $\varepsilon > 0$ be given. Then, by the automorphicity of function f and the definition of \bar{f} , for any sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\|\bar{f}(t, x) - f(t + \alpha_n, x)\| \leq \frac{\varepsilon}{2} \quad \text{and} \quad \|\bar{f}(t, y) - f(t + \alpha_n, y)\| \leq \frac{\varepsilon}{2},$$

for n sufficiently large.

Therefore, we obtain

$$\begin{aligned} \|\bar{f}(t, x) - \bar{f}(t, y)\| &\leq \|\bar{f}(t, x) - f(t + \alpha_n, x)\| + \|\bar{f}(t, y) - f(t + \alpha_n, y)\| \\ &\quad + \|f(t + \alpha_n, x) - f(t + \alpha_n, y)\| \\ &\leq \varepsilon + L\|x - y\|. \end{aligned}$$

Since ε is arbitrary, we get

$$\|\bar{f}(t, x) - \bar{f}(t, y)\| \leq L\|x - y\|,$$

for each $x, y \in X$. \square

The next result will be fundamental to ensure the almost automorphicity of solutions of non-linear dynamic equations. Our proof is inspired by Theorem 2.10 from [3].

Theorem 3.23. *Let \mathbb{T} be an invariant under translations time scale and $f : \mathbb{T} \times X \rightarrow X$ be an almost automorphic function on time scales in t for each x in X and satisfy Lipschitz condition in x uniformly in t , that is*

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|,$$

for all x, y in X . Suppose $\phi : \mathbb{T} \rightarrow X$ is almost automorphic function on time scales, then the function $U : \mathbb{T} \rightarrow X$ defined by $U(t) = f(t, \phi(t))$ is almost automorphic on time scales.

Proof. Since $f, \phi \in AA_{\mathbb{T}}(X)$, then for every sequence (α'_n) in Π , there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that $\lim_{n \rightarrow \infty} f(t + \alpha_n, x) = \bar{f}(t, x)$ for all $t \in \mathbb{T}$, $x \in X$ and $\lim_{n \rightarrow \infty} \bar{f}(t - \alpha_n, x) = f(t, x)$ for all $t \in \mathbb{T}$ and $x \in X$. Also, we have

$$\lim_{n \rightarrow \infty} \phi(t + \alpha_n) = \bar{\phi}(t)$$

is well defined for each $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} \bar{\phi}(t - \alpha_n) = \phi(t)$$

for every $t \in \mathbb{T}$. Since f satisfies the Lipschitz condition in t uniformly in k , then

$$\begin{aligned} & \|f(t + \alpha_n, \phi(t + \alpha_n)) - \bar{f}(t, \bar{\phi}(t))\| \\ & \leq \|f(t + \alpha_n, \phi(t + \alpha_n)) - f(t + \alpha_n, \bar{\phi}(t))\| \\ & \quad + \|f(t + \alpha_n, \bar{\phi}(t)) - \bar{f}(t, \bar{\phi}(t))\| \\ & \leq L\|\phi(t + \alpha_n) - \bar{\phi}(t)\| + \|f(t + \alpha_n, \bar{\phi}(t)) - \bar{f}(t, \bar{\phi}(t))\| \end{aligned}$$

and

$$\begin{aligned} & \|\bar{f}(t - \alpha_n, \bar{\phi}(t - \alpha_n)) - f(t, \phi(t))\| \\ & \leq \|\bar{f}(t - \alpha_n, \bar{\phi}(t - \alpha_n)) - \bar{f}(t - \alpha_n, \phi(t))\| \\ & \quad + \|\bar{f}(t - \alpha_n, \phi(t)) - f(t, \phi(t))\| \\ & \leq L\|\bar{\phi}(t - \alpha_n) - \phi(t)\| + \|\bar{f}(t - \alpha_n, \phi(t)) - f(t, \phi(t))\|. \end{aligned}$$

Notice that if f satisfies Lipschitz condition in x uniformly in t , it is clear by Theorem 3.22 that \bar{f} also satisfies this condition in x uniformly in t for the same constant $L > 0$. Applying limit as $n \rightarrow \infty$ to both inequalities above, we have the desired result. \square

4. Product integration on time scales

In this section, we present some basic concepts concerning product integration on time scales which will be essential to prove our main result. The main reference for this section is [43].

We start this section by presenting a notion of product Δ -integral of a matrix function. For details, see [43].

Given a matrix function $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ and a tagged partition $D \in D(a, b)$, we denote

$$P(A, D) = \prod_{i=m}^1 (I + A(\xi_i) \Delta t_i) = (I + A(\xi_m) \Delta t_m) \cdot (I + A(\xi_1) \Delta t_1).$$

We point out that the order is important since matrix multiplication is usually not commutative.

Now, we present the concept of product Δ -integrable matrix function. See [43].

Definition 4.1. A bounded matrix function $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ is called product Δ -integrable if there exists a matrix $P \in \mathbb{R}^{n \times n}$ with the property that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|P(A, D) - P\| < \varepsilon$ for every $D \in D_\delta(a, b)$. The matrix P is called the *product Δ -integral* of A over $[a, b]_{\mathbb{T}}$ and we write

$$\prod_a^b (I + A(t) \Delta t) = P.$$

If $a = b$, then $\prod_a^a (I + A(t) \Delta t) = I$ for every matrix function $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$.

Now, we present a result which will be essential to our purposes. It can be found in [43, Theorem 2.9].

Theorem 4.2. Let $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ and $t \in \mathbb{T}$, $a \leq t \leq \sigma(t) \leq b$. Then

$$\prod_t^{\sigma(t)} (I + A(s) \Delta s) = I + A(t) \mu(t).$$

The next result brings a property of Riemann Δ -integrable function. It can be found in [43, Theorem 3.7].

Theorem 4.3. Every Riemann Δ -integrable function is product Δ -integrable.

We remind the reader that every rd-continuous function $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^n$ is Riemann Δ -integrable (see [11] and [12]). Thus, we can replace the previous result by the following one.

Theorem 4.4. Every rd-continuous function is product Δ -integrable.

In what follows, we state a property of product Δ integrals. See [43].

Theorem 4.5. *If $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ is Riemann Δ -integrable and $c \in [a, b]_{\mathbb{T}}$, then*

$$\prod_a^b (I + A(t)\Delta t) = \prod_c^b (I + A(t)\Delta t) \cdot \prod_a^c (I + A(t)\Delta t).$$

The next result ensures the continuity of the indefinite product Δ -integral on $[a, b]_{\mathbb{T}}$. See [43, Theorem 4.1].

Theorem 4.6. *If $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ is Riemann Δ -integrable, then the indefinite product Δ -integral*

$$Y(t) = \prod_a^t (I + A(s)\Delta s), \quad t \in [a, b]_{\mathbb{T}},$$

is continuous on $[a, b]_{\mathbb{T}}$.

Theorem 4.7. (See [43, Theorem 5.5].) *If $A : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^{n \times n}$ is a regressive Riemann Δ -integrable function, then $\prod_a^b (I + A(t)\Delta t)$ is a nonsingular matrix.*

If $a < b$, we define $\prod_a^b (I + A(t)\Delta t) = (\prod_a^b (I + A(t)\Delta t))^{-1}$ provided the right-hand side exists. See [43, Definition 5.6].

Finally, we present a result which will be fundamental to prove our main result. It can be found in [43, Theorem 5.7].

Theorem 4.8. *If $A : \mathbb{T} \rightarrow \mathbb{R}^{n \times n}$ is a regressive rd-continuous function and $t_0 \in \mathbb{T}$, then the function*

$$Y(t) = \prod_{t_0}^t (I + A(s)\Delta s), \quad t \in \mathbb{T},$$

represents the unique solution of the dynamic equation $Y^\Delta(t) = A(t)Y(t)$ such that $Y(t_0) = I$.

5. Almost automorphic solutions of first-order linear dynamic equations on time scales

In this section, our goal is to prove the existence of an almost automorphic solution of first-order linear dynamic equation on time scales given by

$$x^\Delta(t) = A(t)x(t) + f(t) \tag{5.1}$$

where $A : \mathbb{T} \rightarrow \mathbb{R}^{n \times n}$, $f : \mathbb{T} \rightarrow \mathbb{R}^n$ and its associated homogeneous equation:

$$x^\Delta(t) = A(t)x(t). \tag{5.2}$$

Throughout this section, we assume that $A(t)$ is almost automorphic on \mathbb{T} , which means that given a sequence $(\alpha'_n) \in \mathcal{I}$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n \rightarrow \infty} A(t + \alpha_n) = \bar{A}(t) \quad (5.3)$$

exists and is well defined for every $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} \bar{A}(t - \alpha_n) = A(t) \quad (5.4)$$

for every $t \in \mathbb{T}$.

Also, consider the following linear dynamic system

$$x^\Delta(t) = \bar{A}(t)x(t). \quad (5.5)$$

Before to proceed, we present a result which will be fundamental to our objectives. It can be found in [41, Lemma 2.9] for the case $\mathbb{T} = \mathbb{Z}$. We give the proof here for the general case.

Lemma 5.1. *Let \mathbb{T} be an invariant under translations time scale and $A(t)$ be an almost automorphic and nonsingular matrix on \mathbb{T} . Also, suppose that the set $\{A^{-1}(t)\}_{t \in \mathbb{T}}$ is bounded. Then $A^{-1}(t)$ is almost automorphic on \mathbb{T} , that is, for every sequence (α'_n) on Π , there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that*

$$\lim_{n \rightarrow \infty} A^{-1}(t + \alpha_n) =: \bar{A}^{-1}(t) \quad (5.6)$$

is well defined for each $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} \bar{A}^{-1}(t - \alpha_n) = A^{-1}(t) \quad (5.7)$$

for each $t \in \mathbb{T}$.

Proof. Let (α'_n) be a sequence on Π . Since $A(t)$ is almost automorphic on time scales, there exists a subsequence (α_n) such that

$$\lim_{n \rightarrow \infty} A(t + \alpha_n) =: \bar{A}(t)$$

is well defined for each $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} \bar{A}(t - \alpha_n) = A(t)$$

for each $t \in \mathbb{T}$.

Fix $t \in \mathbb{T}$ and define $A_n := A(t + \alpha_n)$, $n \in \mathbb{N}$. By hypothesis, the set $\{A_n^{-1}\}_{n \in \mathbb{N}}$ is bounded. Using the identity

$$A_n^{-1} - A_m^{-1} = A_n^{-1}(A_m - A_n)A_m^{-1}$$

and the fact that $\{A_n\}$ is a Cauchy sequence, it follows that $\{A_n^{-1}\}$ is a Cauchy sequence. Hence, there exists a matrix T (for each $t \in \mathbb{T}$ fixed) such that

$$A_n^{-1} \rightarrow T(t).$$

Taking the limit of $A_n A_n^{-1} = A_n^{-1} A_n = I$, where I denotes the identity matrix, we obtain that $\bar{A}(t)$ is invertible and $\bar{A}^{-1}(t) = T(t)$ for each $t \in \mathbb{T}$. Since the map $A \rightarrow A^{-1}$ is continuous on the set of nonsingular matrices, it follows that

$$\lim_{n \rightarrow \infty} A^{-1}(t + \alpha_n) = \bar{A}^{-1}(t)$$

for each $t \in \mathbb{T}$. Analogously, one can prove that

$$\lim_{n \rightarrow \infty} \bar{A}^{-1}(t - \alpha_n) = A^{-1}(t),$$

for each $t \in \mathbb{T}$. \square

As an immediate consequence, we obtain the following result for a particular case.

Corollary 5.2. *Let \mathbb{T} be an invariant under translations time scale and $A(t)$ be an almost automorphic and regressive matrix on \mathbb{T} . Also, suppose that the set $\{(I + A(t)\mu(t))^{-1}\}_{t \in \mathbb{T}}$ is bounded. Then $(I + A(t)\mu(t))^{-1}$ is almost automorphic on \mathbb{T} , that is, for every sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that*

$$\lim_{n \rightarrow \infty} (I + A(t + \alpha_n)\mu(t + \alpha_n))^{-1} =: (I + \bar{A}(t)\bar{\mu}(t))^{-1} \quad (5.8)$$

is well defined for each $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} (I + \bar{A}(t - \alpha_n)\bar{\mu}(t - \alpha_n))^{-1} = (I + A(t)\mu(t))^{-1} \quad (5.9)$$

for each $t \in \mathbb{T}$.

Proof. Denote $B(t) := (I + A(t)\mu(t))$, for every $t \in \mathbb{T}$. Since $A(t)$ and $\mu(t)$ are almost automorphic functions, for every sequence (α'_n) on Π , there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n \rightarrow \infty} A(t + \alpha_n) = \bar{A}(t) \quad \text{and} \quad \lim_{n \rightarrow \infty} \mu(t - \alpha_n) = \bar{\mu}(t)$$

are well defined and exist for every $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} \bar{A}(t - \alpha_n) = A(t) \quad \text{and} \quad \lim_{n \rightarrow \infty} \bar{\mu}(t - \alpha_n) = \mu(t)$$

for every $t \in \mathbb{T}$. Thus, it follows that

$$\lim_{n \rightarrow \infty} B(t + \alpha_n) = \lim_{n \rightarrow \infty} (I + A(t + \alpha_n)\mu(t + \alpha_n)) = I + \bar{A}(t)\bar{\mu}(t) := \bar{B}(t)$$

for every $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} \bar{B}(t - \alpha_n) = \lim_{n \rightarrow \infty} (I + \bar{A}(t - \alpha_n)\bar{\mu}(t - \alpha_n)) = I + A(t)\mu(t) = B(t).$$

Therefore, $B(t)$ is almost automorphic on \mathbb{T} . Also, since $A(t)$ is a regressive matrix, it follows that $B(t)$ is nonsingular on \mathbb{T} . By hypothesis, $\{B^{-1}(t)\}_{t \in \mathbb{T}}$ is bounded. Thus, all the hypotheses of [Lemma 5.1](#) are satisfied. As a consequence, we obtain that $B^{-1}(t) = (I + A(t)\mu(t))^{-1}$ is almost automorphic on \mathbb{T} , that is, for every sequence (α'_n) on Π , there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n \rightarrow \infty} B^{-1}(t + \alpha_n) =: \bar{B}^{-1}(t)$$

is well defined and exists for each $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} \bar{B}^{-1}(t - \alpha_n) = B^{-1}(t)$$

for each $t \in \mathbb{T}$. It implies that

$$\lim_{n \rightarrow \infty} (I + A(t + \alpha_n)\mu(t + \alpha_n))^{-1} =: (I + \bar{A}(t)\bar{\mu}(t))^{-1}$$

is well defined for each $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} (I + \bar{A}(t - \alpha_n)\bar{\mu}(t - \alpha_n))^{-1} = (I + A(t)\mu(t))^{-1}$$

for each $t \in \mathbb{T}$ and the result follows as well. \square

Now, we present some auxiliary results lemma which will be essential to our purposes.

Lemma 5.3. *Let \mathbb{T} be an invariant under translations and $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n})$ is almost automorphic and nonsingular on \mathbb{T} and $\{A^{-1}(t)\}_{t \in \mathbb{T}}$ and $\{(I + \mu(t)A(t))^{-1}\}_{t \in \mathbb{T}}$ are bounded on \mathbb{T} . Then for every sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that*

$$\lim_{n \rightarrow \infty} X(t + \alpha_n)X^{-1}(s + \alpha_n) = W(t, s)$$

for every $s, t \in \mathbb{T}$, $s < t$ and

$$\lim_{n \rightarrow \infty} W(t - \alpha_n, s - \alpha_n) = X(t)X^{-1}(s),$$

for every $s, t \in \mathbb{T}$ and $s < t$, where $X(t)$ is the fundamental matrix of [\(5.2\)](#) and $W(t, s) := \prod_s^t (I + \bar{A}(\tau)\Delta\tau)$.

Proof. Since $X(t)$ is the fundamental matrix of [\(5.2\)](#), we obtain by [Theorem 4.8](#)

$$X(t) = \prod_{t_0}^t (I + A(\tau)\Delta\tau)$$

which implies the following

$$X^{-1}(t) = (X(t))^{-1} = \left(\prod_{t_0}^t (I + A(\tau)\Delta\tau) \right)^{-1}.$$

Notice that, by [Theorem 4.7](#), $\Pi_{t_0}^t(I + A(\tau)\Delta\tau)$ is a nonsingular matrix, since $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n})$.

By the almost automorphicity of $A(t)$, we obtain that for every sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\begin{aligned} X(t + \alpha_n)X^{-1}(s + \alpha_n) &= \prod_{t_0}^{t+\alpha_n} (I + A(\tau)\Delta\tau) \left(\prod_{t_0}^{s+\alpha_n} (I + A(\tau)\Delta\tau) \right)^{-1} \\ &= \prod_{t_0}^{t+\alpha_n} (I + A(\tau)\Delta\tau) \prod_{s+\alpha_n}^{t_0} (I + A(\tau)\Delta\tau) \\ &= \prod_{s+\alpha_n}^{t+\alpha_n} (I + A(\tau)\Delta\tau) = \prod_s^t (I + A(\tau + \alpha_n)\Delta\tau), \end{aligned}$$

by [Theorem 4.5](#). Applying the limit in both sides, we obtain

$$\lim_{n \rightarrow \infty} X(t + \alpha_n)X^{-1}(s + \alpha_n) = \lim_{n \rightarrow \infty} \prod_s^t (I + A(\tau + \alpha_n)\Delta\tau) = \prod_s^t (I + \bar{A}(\tau)\Delta\tau) := W(t, s),$$

since the product Δ -integral is a continuous function, by [Theorem 4.6](#).

We also point out that \bar{A} is Riemann Δ -integrable, since \bar{A} is a bounded function (see [\[12\]](#)). And thus, by [Theorem 4.3](#), \bar{A} is also product Δ -integrable.

Analogously, one can prove that

$$\lim_{n \rightarrow \infty} W(t - \alpha_n, s - \alpha_n) = X(t)X^{-1}(s). \quad \square$$

Remark 5.4. It is clear that from the previous result, by the same hypothesis, we obtain as a consequence the following

$$\lim_{n \rightarrow \infty} X(t + \alpha_n)PX^{-1}(s + \alpha_n) = PW(t, s)$$

and

$$\lim_{n \rightarrow \infty} PW(t - \alpha_n, s - \alpha_n) = X(t)PX^{-1}(s),$$

for a projection P described in [Definition 2.12](#) and also, we have

$$\lim_{n \rightarrow \infty} X(t + \alpha_n)(I - P)X^{-1}(s + \alpha_n) = (I - P)W(t, s)$$

and

$$\lim_{n \rightarrow \infty} (I - P)W(t - \alpha_n, s - \alpha_n) = X(t)(I - P)X^{-1}(s).$$

Lemma 5.5. *Let \mathbb{T} be an invariant under translations time scale, $A(t) \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n})$ be almost automorphic and nonsingular on \mathbb{T} and the sets $\{A^{-1}(t)\}_{t \in \mathbb{T}}$ and $\{(I + \mu(t)A(t))^{-1}\}_{t \in \mathbb{T}}$ are bounded on \mathbb{T} . Also, suppose the system (5.2) has an exponential dichotomy with positive constants K and γ . Then, given a sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that*

$$\lim_{n \rightarrow \infty} X(t + \alpha_n)PX^{-1}(\sigma(s + \alpha_n)) = PW(t, s)(I + \bar{A}(s)\bar{\mu}(s))^{-1} := Y(t, s) \quad (5.10)$$

exists and is well defined for $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} Y(t - \alpha_n, s - \alpha_n) = X(t)PX^{-1}(\sigma(s)), \quad (5.11)$$

where $\bar{\mu}(t)$ is given by (3.2) and (3.3), $X(t)$ is the fundamental matrix of (5.2) and $W(t, s) = \prod_s^t (I + \bar{A}(\tau))$. Similarly, we have that given a sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} X(t + \alpha_n)(I - P)X^{-1}(\sigma(s + \alpha_n)) &= (I - P)W(t, s)(I + \bar{A}(s)\bar{\mu}(s))^{-1} \\ &:= Z(t, s) \end{aligned} \quad (5.12)$$

exists and is well defined for $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} Z(t - \alpha_n, s - \alpha_n) = X(t)(I - P)X^{-1}(\sigma(s)). \quad (5.13)$$

Proof. Since $X(t)$ is the fundamental matrix of (5.2), we have for $s < t$

$$\begin{aligned} X(t + \alpha_n)PX^{-1}(\sigma(s + \alpha_n)) &= X(t + \alpha_n)PX^{-1}(s + \alpha_n)X(s + \alpha_n)X^{-1}(\sigma(s + \alpha_n)) \\ &= X(t + \alpha_n)PX^{-1}(s + \alpha_n) \left(\prod_{\sigma(s + \alpha_n)}^{s + \alpha_n} (I + A(\tau)\Delta\tau) \right) \\ &= X(t + \alpha_n)PX^{-1}(s + \alpha_n) \left(\prod_{s + \alpha_n}^{\sigma(s + \alpha_n)} (I + A(\tau)\Delta\tau) \right)^{-1} \\ &= X(t + \alpha_n)PX^{-1}(s + \alpha_n)(I + A(s + \alpha_n)\mu(s + \alpha_n))^{-1}. \end{aligned}$$

Applying the limit as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} X(t + \alpha_n)PX^{-1}(\sigma(s + \alpha_n)) = PW(t, s)(I + \bar{A}(s)\bar{\mu}(s))^{-1} := Y(t, s),$$

by Remark 5.4, the automorphicity of μ and using the fact that the map $I + A(s)\mu(s) \mapsto (I + A(s)\mu(s))^{-1}$ is continuous on the set of regressive matrices. Notice that $(I + \bar{A}(s)\bar{\mu}(s))^{-1}$ is well defined, by Corollary 5.2. We remind that the projection P commutes with $X(t)$ (see Definition 2.12).

Reciprocally, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} Y(t - \alpha_n, s - \alpha_n) &:= \lim_{n \rightarrow \infty} PW(t - \alpha_n, s - \alpha_n)(I + \bar{A}(s - \alpha_n)\bar{\mu}(s - \alpha_n))^{-1} \\ &= X(t)PX^{-1}(s)(I + A(s)\mu(s))^{-1} \\ &= X(t)PX^{-1}(s)\left(\prod_s^{\sigma(s)}(I + A(\tau)\Delta\tau)\right)^{-1} \\ &= X(t)PX^{-1}(s)\prod_{\sigma(s)}^s(I + A(\tau)\Delta\tau) \\ &= X(t)PX^{-1}(s)X(s)X^{-1}(\sigma(s)) = X(t)PX^{-1}(\sigma(s)). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} X(t + \alpha_n)(I - P)X^{-1}(\sigma(s + \alpha_n)) \\ &= X(t + \alpha_n)(I - P)X^{-1}(s + \alpha_n)X(s + \alpha_n)X^{-1}(\sigma(s + \alpha_n)) \\ &= X(t + \alpha_n)(I - P)X^{-1}(s + \alpha_n)\left(\prod_{\sigma(s + \alpha_n)}^{s + \alpha_n}(I + A(\tau)\Delta\tau)\right) \\ &= X(t + \alpha_n)(I - P)X^{-1}(s + \alpha_n)\left(\prod_{s + \alpha_n}^{\sigma(s + \alpha_n)}(I + A(\tau)\Delta\tau)\right)^{-1} \\ &= X(t + \alpha_n)(I - P)X^{-1}(s + \alpha_n)(I + A(s + \alpha_n)\mu(s + \alpha_n))^{-1}. \end{aligned}$$

Applying the limit as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} X(t + \alpha_n)(I - P)X^{-1}(\sigma(s + \alpha_n)) = (I - P)W(t, s)(I + \bar{A}(s)\bar{\mu}(s))^{-1} := Z(t, s),$$

by Remark 5.4, the automorphicity of μ and using the fact that the map $I + A(s)\mu(s) \mapsto (I + A(s)\mu(s))^{-1}$ is continuous on the set of regressive matrices.

Reciprocally, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} Z(t - \alpha_n, s - \alpha_n) &:= \lim_{n \rightarrow \infty} (I - P)W(t - \alpha_n, s - \alpha_n)(I + \bar{A}(s - \alpha_n)\bar{\mu}(s - \alpha_n))^{-1} \\ &= X(t)(I - P)X^{-1}(s)(I + A(s)\mu(s))^{-1} \\ &= X(t)(I - P)X^{-1}(s)\left(\prod_s^{\sigma(s)}(I + A(\tau)\Delta\tau)\right)^{-1} \end{aligned}$$

$$\begin{aligned}
&= X(t)(I - P)X^{-1}(s) \prod_{\sigma(s)}^s (I + A(\tau)\Delta\tau) \\
&= X(t)(I - P)X^{-1}(s)X(s)X^{-1}(\sigma(s)) = X(t)(I - P)X^{-1}(\sigma(s)),
\end{aligned}$$

and the result follows as well. \square

Now, we present our main result in this section. The next result ensures that the system (5.1) has an almost automorphic solution.

Theorem 5.6. *Let \mathbb{T} be an invariant under translations time scale and $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n})$ be almost automorphic and nonsingular on \mathbb{T} and $\{A^{-1}(t)\}_{t \in \mathbb{T}}$ and $\{(I + \mu(t)A(t))^{-1}\}_{t \in \mathbb{T}}$ are bounded. Also, suppose Eq. (5.2) admits an exponential dichotomy with positive constants K and γ and $f \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}^n)$ is an almost automorphic function on time scales. Then Eq. (5.1) has an almost automorphic solution.*

Proof. By Theorem 2.15, we have that the following function

$$x(t) = \int_{-\infty}^t X(t)PX^{-1}(\sigma(s))f(s)\Delta s - \int_t^{+\infty} X(t)(I - P)X^{-1}(\sigma(s))f(s)\Delta s \quad (5.14)$$

is a bounded solution of (2.1). It remains to prove that $x : \mathbb{T} \rightarrow \mathbb{R}^n$ is an almost automorphic function.

By the automorphicity of functions $A(t)$, $f(t)$ and $\mu(t)$, it follows that for every sequence (α'_n) on \mathbb{T} , there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n \rightarrow \infty} A(t + \alpha_n) = \bar{A}(t), \quad \lim_{n \rightarrow \infty} f(t + \alpha_n) = \bar{f}(t) \quad \text{and} \quad \lim_{n \rightarrow \infty} \mu(t + \alpha_n) = \bar{\mu}(t)$$

are well defined and exist for every $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} \bar{A}(t - \alpha_n) = A(t), \quad \lim_{n \rightarrow \infty} \bar{f}(t - \alpha_n) = f(t) \quad \text{and} \quad \lim_{n \rightarrow \infty} \bar{\mu}(t - \alpha_n) = \mu(t)$$

for every $t \in \mathbb{T}$.

Let us denote

$$M(t) = \int_{-\infty}^t X(t)PX^{-1}(\sigma(s))f(s)\Delta s$$

and

$$\bar{M}(t) = \int_{-\infty}^t Y(t, s)\bar{f}(s)\Delta s,$$

where $Y(t, s)$ is given by (5.10).

Then, we have

$$\begin{aligned}
 & \|M(t + \alpha_n) - \bar{M}(t)\| \\
 &= \left\| \int_{-\infty}^{t+\alpha_n} X(t + \alpha_n) P X^{-1}(\sigma(s)) f(s) \Delta s - \int_{-\infty}^t Y(t, s) \bar{f}(s) \Delta s \right\| \\
 &= \left\| \int_{-\infty}^t X(t + \alpha_n) P X^{-1}(\sigma(s + \alpha_n)) f(s + \alpha_n) \Delta s - \int_{-\infty}^t Y(t, s) \bar{f}(s) \Delta s \right\| \\
 &\leq \left\| \int_{-\infty}^t X(t + \alpha_n) P X^{-1}(\sigma(s + \alpha_n)) f(s + \alpha_n) \Delta s \right. \\
 &\quad \left. - \int_{-\infty}^t X(t + \alpha_n) P X^{-1}(\sigma(s + \alpha_n)) \bar{f}(s) \Delta s \right\| \\
 &\quad + \left\| \int_{-\infty}^t X(t + \alpha_n) P X^{-1}(\sigma(s + \alpha_n)) \bar{f}(s) \Delta s - \int_{-\infty}^t Y(t, s) \bar{f}(s) \Delta s \right\| \\
 &= \left\| \int_{-\infty}^t X(t + \alpha_n) P X^{-1}(\sigma(s + \alpha_n)) [f(s + \alpha_n) - \bar{f}(s)] \Delta s \right\| \\
 &\quad + \left\| \int_{-\infty}^t [X(t + \alpha_n) P X^{-1}(\sigma(s + \alpha_n)) - Y(t, s)] \bar{f}(s) \Delta s \right\|.
 \end{aligned}$$

Applying limit as $n \rightarrow \infty$ and using the fact that \bar{f} is a bounded function (this fact follows by automorphy of f) and the exponential dichotomy of Eq. (5.2), we obtain

$$\lim_{n \rightarrow \infty} M(t + \alpha_n) = \bar{M}(t)$$

for each $t \in \mathbb{T}$. Similarly, we can prove

$$\lim_{n \rightarrow \infty} \bar{M}(t - \alpha_n) = M(t)$$

for each $t \in \mathbb{T}$.

Let us denote

$$N(t) = \int_t^{+\infty} X(t)(I - P)X^{-1}(\sigma(s)) f(s) \Delta s$$

and

$$\bar{N}(t) = \int_t^{+\infty} Z(t, s) \bar{f}(s) \Delta s,$$

where $Z(t, s)$ is given by (5.12).

Then, the same way as before, one can prove that given a sequence $(\alpha'_n) \in \mathbb{T}$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n \rightarrow \infty} N(t + \alpha_n) = \bar{N}(t)$$

for every $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} \bar{N}(t - \alpha_n) = N(t)$$

for each $t \in \mathbb{T}$.

Now, define $\bar{x}(t) = \bar{M}(t) + \bar{N}(t)$, then using the definition of x from (5.14), we obtain as an immediate consequence that

$$\lim_{n \rightarrow \infty} x(t + \alpha_n) = \bar{x}(t)$$

is well defined for every $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} \bar{x}(t - \alpha_n) = x(t)$$

for each $t \in \mathbb{T}$.

Thus, x is an almost automorphic function and we get the desired result. \square

Remark 5.7. It is clear that the previous theorem remains valid for linear nabla dynamic equations on time scales. In the other words, one can prove analogously that the nabla dynamic equation

$$x^\nabla(t) = A(t)x(t) + f(t), \quad (5.15)$$

where $A : \mathbb{T} \rightarrow \mathbb{R}^{n \times n}$ and $f : \mathbb{T} \rightarrow \mathbb{R}^n$, has an almost automorphic solution, under similar conditions to the ones presented in Theorem 5.6.

Choosing $\mathbb{T} = \mathbb{Z}$ in Theorem 5.6, we obtain a result for difference equations. It is the content of the next result.

Corollary 5.8. Let $A : \mathbb{Z} \rightarrow \mathbb{R}^{n \times n}$ be almost automorphic, regulated and nonsingular matrix function. Also, suppose that $\{A^{-1}(k)\}_{k \in \mathbb{Z}}$ is bounded. Moreover, assume that $(I + A(t))$ is nonsingular matrix function and $\{(I + A(k))^{-1}\}_{k \in \mathbb{Z}}$ is bounded, the equation

$$x(k+1) = A(k)x(k)$$

admits an exponential dichotomy with positive constants K and γ and the function $f : \mathbb{Z} \rightarrow \mathbb{R}^n$ is almost automorphic and regulated. Then the equation

$$x(k+1) = A(k)x(k) + f(k)$$

has an almost automorphic solution.

Similarly, we can take $\mathbb{T} = h\mathbb{Z}$ in [Theorem 5.6](#) and obtain an interesting result for a different type of difference equations.

Corollary 5.9. *Let $A : h\mathbb{Z} \rightarrow \mathbb{R}^{n \times n}$ be almost automorphic, regulated and nonsingular matrix function. Also, suppose that $\{A^{-1}(k)\}_{k \in h\mathbb{Z}}$ is bounded. Moreover, assume that $(I + hA(t))$ is nonsingular matrix function and $\{(I + hA(k))^{-1}\}_{k \in h\mathbb{Z}}$ is bounded, the equation*

$$x(k+h) = A(k)x(k)$$

admits an exponential dichotomy with positive constants K and γ and the function $f : h\mathbb{Z} \rightarrow \mathbb{R}^n$ is almost automorphic and regulated. Then the equation

$$x(k+h) = A(k)x(k) + f(k)$$

has an almost automorphic solution.

Now, taking $\mathbb{T} = \mathbb{R}$ in [Theorem 5.6](#), one can obtain a result for ordinary differential equations.

Corollary 5.10. *Let $A : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ be almost automorphic, continuous and nonsingular matrix function and $\{A^{-1}(t)\}_{t \in \mathbb{R}}$ is bounded. Assume also the equation*

$$\dot{x}(t) = A(t)x(t)$$

admits an exponential dichotomy with positive constants K and γ and the function $f : \mathbb{R} \rightarrow \mathbb{R}^n$ is almost automorphic and continuous. Then the equation

$$\dot{x}(t) = A(t)x(t) + f(t)$$

has an almost automorphic solution.

Finally, we choose $\mathbb{T} = \mathbb{P}_{a, \cos a}$ and obtain an interesting result. We do not know any result concerning almost automorphic treating about this case.

Corollary 5.11. *Let $A : \mathbb{P}_{a, \cos a} \rightarrow \mathbb{R}^{n \times n}$, for $0 < a < \frac{\pi}{2}$ be almost automorphic and, for every $k \in \mathbb{Z}$, be continuous at $[k(a + \cos a), k(a + \cos a) + a)$, regulated at $k(a + \cos a) + a$ and a nonsingular matrix function. Also, suppose that $\{A^{-1}(t)\}_{t \in \mathbb{P}_{a, \cos a}}$ is bounded. Moreover, assume that $(I + (\cos t)A(t))$ is nonsingular matrix function and $\{(I + (\cos t)A(t))^{-1}\}_{t \in \mathbb{P}_{a, \cos a}}$ is bounded and the equation*

$$x^\Delta(t) = A(t)x(t)$$

admits an exponential dichotomy with positive constants K and γ and the function $f : \mathbb{P}_{a, \cos a} \rightarrow \mathbb{R}^n$ is almost automorphic and, for every $k \in \mathbb{N}$, be continuous at $[k(a + \cos a), k(a + \cos a) + a)$ and regulated at $k(a + \cos a) + a$. Then the equation

$$x^\Delta(t) = A(t)x(t) + f(t)$$

has an almost automorphic solution.

6. Almost automorphic solutions for semilinear dynamic equations on time scales

In this section, consider the following semilinear dynamic equation

$$x^\Delta(t) = A(t)x(t) + f(t, x) \quad (6.1)$$

where $A : \mathbb{T} \rightarrow \mathbb{R}^{n \times n}$ and $f : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and its associated homogeneous equation:

$$x^\Delta(t) = A(t)x(t). \quad (6.2)$$

Also, consider the following linear dynamic system

$$x^\Delta(t) = \bar{A}(t)x(t), \quad (6.3)$$

where $\bar{A}(t)$ is given by (5.3) and (5.4).

Now, we introduce a definition of solution of (6.1) in a strict sense. Here, we will restrict ourselves for this concept of solution for (6.1).

Definition 6.1. We say that $x : \mathbb{T} \rightarrow \mathbb{R}^n$ is a *solution* of (6.1) if x satisfies the following equation

$$\begin{aligned} x(t) = & \int_{-\infty}^t X(t)P X^{-1}(\sigma(s))f(s, x(s))\Delta s \\ & - \int_t^{+\infty} X(t)(I - P)X^{-1}(\sigma(s))f(s, x(s))\Delta s. \end{aligned} \quad (6.4)$$

Remark 6.2. We point out that the previous definition makes sense. In fact, suppose $x(t)$ satisfies Eq. (6.4), then

$$\begin{aligned} & x^\Delta(t) - A(t)x(t) \\ &= X^\Delta(t) \int_{-\infty}^t P X^{-1}(\sigma(s))f(s, x(s))\Delta s + X(\sigma(t))P X^{-1}(\sigma(t))f(t, x(t)) \\ & \quad - X^\Delta(t) \int_t^{+\infty} (I - P)X^{-1}(\sigma(s))f(s, x(s))\Delta s + X(\sigma(t))(I - P)X^{-1}(\sigma(t))f(t, x(t)) \end{aligned}$$

$$\begin{aligned}
& -A(t)X(t) \int_{-\infty}^t PX^{-1}(\sigma(s))f(s, x(s))\Delta s \\
& + A(t)X(t) \int_t^{+\infty} (I - P)X^{-1}(\sigma(s))f(s, x(s))\Delta s \\
& = X(\sigma(t))(P + I - P)X^{-1}(\sigma(t))f(t, x(t)) = f(t, x(t))
\end{aligned}$$

which implies

$$x^\Delta(t) = A(t)x(t) + f(t, x(t)).$$

The proof of [Remark 6.2](#) follows analogously the proof of Lemma 2.13 from [\[37\]](#). We reproduce it here for reader's convenience.

In the sequel, we present an existence and uniqueness result of an almost automorphic solution of [\(6.1\)](#).

Theorem 6.3. *Let \mathbb{T} be an invariant under translations time scale and $f \in \mathcal{C}_{rd}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ be almost automorphic with respect to the first variable. Assume that $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n})$ is almost automorphic and nonsingular matrix function, the sets $\{A^{-1}(t)\}_{t \in \mathbb{T}}$ and $\{(I + \mu(t)A(t))^{-1}\}_{t \in \mathbb{T}}$ are bounded. Suppose also [Eq. \(6.2\)](#) admits an exponential dichotomy on \mathbb{T} with positive constants K and γ and the following condition is fulfilled:*

(i) *There exists a constant $0 < L < \frac{\gamma}{2K(2+\tilde{\mu}\gamma)}$ such that*

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|,$$

for every $x, y \in \mathbb{R}^n$ and $t \in \mathbb{T}$, where $\tilde{\mu} = \sup_{t \in \mathbb{T}} |\mu(t)|$.

Then, the system [\(6.1\)](#) has a unique solution which is almost automorphic.

Proof. Define an operator $T : AA_{\mathbb{T}}(\mathbb{R}^n) \rightarrow AA_{\mathbb{T}}(\mathbb{R}^n)$ as follows:

$$(Tu)(t) = \int_{-\infty}^t X(t)PX^{-1}(\sigma(s))f(s, u(s))\Delta s - \int_t^{+\infty} X(t)(I - P)X^{-1}(\sigma(s))f(s, u(s))\Delta s,$$

for all $u \in AA_{\mathbb{T}}(\mathbb{R}^n)$.

Now, let us prove that $Tu \in AA_{\mathbb{T}}(\mathbb{R}^n)$. Since f satisfies the Lipschitz condition, we obtain by [Theorem 3.23](#) that $f(\cdot, u(\cdot))$ is almost automorphic, using the fact that $f, u \in AA_{\mathbb{T}}(\mathbb{R}^n)$. Since $u \in AA_{\mathbb{T}}(\mathbb{R}^n)$, then for every sequence $(\alpha''_n) \in \Pi$, there exists a subsequence $(\alpha'_n) \subset (\alpha''_n)$ such that

$$\lim_{n \rightarrow \infty} u(t + \alpha'_n) = \bar{u}(t)$$

is well defined for every $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} \bar{u}(t - \alpha_n) = u(t),$$

for each $t \in \mathbb{T}$.

Moreover, by the automorphicity of f , we obtain that there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n \rightarrow \infty} f(t + \alpha'_n, u) = \bar{f}(t, u)$$

is well defined for every $t \in \mathbb{T}$, $u \in \mathbb{R}^n$ and

$$\lim_{n \rightarrow \infty} \bar{f}(t - \alpha_n, u) = f(t, u),$$

for each $t \in \mathbb{T}$.

By [Lemma 5.5](#), we obtain

$$\lim_{n \rightarrow \infty} X(t + \alpha_n) P X^{-1}(\sigma(s + \alpha_n)) = Y(t, s)$$

exists and is well defined for $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} Y(t - \alpha_n, s - \alpha_n) = X(t) P X^{-1}(\sigma(s)),$$

where $X(t)$ is the fundamental matrix of [\(6.2\)](#) and $Y(t, s)$ is given by [\(5.10\)](#).

Similarly, by [Lemma 5.5](#), we get

$$\lim_{n \rightarrow \infty} X(t + \alpha_n) (I - P) X^{-1}(\sigma(s + \alpha_n)) = Z(t, s)$$

exists and is well defined for $t \in \mathbb{T}$ and

$$\lim_{n \rightarrow \infty} Z(t - \alpha_n, s - \alpha_n) = X(t) (I - P) X^{-1}(\sigma(s)),$$

where $X(t)$ is the fundamental matrix of [\(6.2\)](#) and $Z(t, s)$ is given by [\(5.12\)](#).

Let us define the following function:

$$h(t) = \int_{-\infty}^t Y(t, s) \bar{f}(s, \bar{u}(s)) \Delta s - \int_t^{+\infty} Z(t, s) \bar{f}(s, \bar{u}(s)) \Delta s,$$

for every $t \in \mathbb{T}$.

Then, we obtain

$$\begin{aligned} & \| (Tu)(t + \alpha_n) - h(t) \| \\ & \leq \left\| \int_{-\infty}^{t + \alpha_n} X(t + \alpha_n) P X^{-1}(\sigma(s)) f(s, u(s)) \Delta s - \int_{-\infty}^t Y(t, s) \bar{f}(s, \bar{u}(s)) \Delta s \right\| \end{aligned}$$

$$\begin{aligned}
& + \left\| \int_{t+\alpha_n}^{+\infty} X(t+\alpha_n)(I-P)X^{-1}(\sigma(s))f(s, u(s))\Delta s - \int_t^{+\infty} Z(t, s)\bar{f}(s, \bar{u}(s))\Delta s \right\| \\
& = \left\| \int_{-\infty}^t X(t+\alpha_n)PX^{-1}(\sigma(s+\alpha_n))f(s+\alpha_n, u(s+\alpha_n))\Delta s - \int_{-\infty}^t Y(t, s)\bar{f}(s, \bar{u}(s))\Delta s \right\| \\
& \quad + \left\| \int_t^{+\infty} X(t+\alpha_n)(I-P)X^{-1}(\sigma(s+\alpha_n))f(s+\alpha_n, u(s+\alpha_n))\Delta s \right. \\
& \quad \left. - \int_t^{+\infty} Z(t, s)\bar{f}(s, \bar{u}(s))\Delta s \right\| \\
& \leq \int_{-\infty}^t \|X(t+\alpha_n)PX^{-1}(\sigma(s+\alpha_n)) - Y(t, s)\| \|\bar{f}(s, \bar{u}(s))\| \Delta s \\
& \quad + \int_{-\infty}^t \|X(t+\alpha_n)PX^{-1}(\sigma(s+\alpha_n))\| \|f(s+\alpha_n, u(s+\alpha_n)) - \bar{f}(s, \bar{u}(s))\| \Delta s \\
& \quad + \int_t^{+\infty} \|X(t+\alpha_n)(I-P)X^{-1}(\sigma(s+\alpha_n)) - Z(t, s)\| \|\bar{f}(s, \bar{u}(s))\| \Delta s \\
& \quad + \int_t^{+\infty} \|X(t+\alpha_n)(I-P)X^{-1}(\sigma(s+\alpha_n))\| \|f(s+\alpha_n, u(s+\alpha_n)) - \bar{f}(s, \bar{u}(s))\| \Delta s.
\end{aligned}$$

Applying the limit as $n \rightarrow \infty$ in both sides, we obtain

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \|(Tu)(t+\alpha_n) - h(t)\| \\
& \leq \lim_{n \rightarrow \infty} \int_{-\infty}^t \|X(t+\alpha_n)PX^{-1}(\sigma(s+\alpha_n)) - Y(t, s)\| \|\bar{f}(s, \bar{u}(s))\| \Delta s \\
& \quad + \lim_{n \rightarrow \infty} \int_{-\infty}^t \|X(t+\alpha_n)PX^{-1}(\sigma(s+\alpha_n))\| \|f(s+\alpha_n, u(s+\alpha_n)) - \bar{f}(s, \bar{u}(s))\| \Delta s \\
& \quad + \lim_{n \rightarrow \infty} \int_t^{+\infty} \|X(t+\alpha_n)(I-P)X^{-1}(\sigma(s+\alpha_n)) - Z(t, s)\| \|\bar{f}(s, \bar{u}(s))\| \Delta s \\
& \quad + \lim_{n \rightarrow \infty} \int_t^{+\infty} \|X(t+\alpha_n)(I-P)X^{-1}(\sigma(s+\alpha_n))\| \|f(s+\alpha_n, u(s+\alpha_n)) - \bar{f}(s, \bar{u}(s))\| \Delta s.
\end{aligned}$$

By the exponential dichotomy and the almost automorphicity of f , we obtain

$$\lim_{n \rightarrow \infty} Tu(t + \alpha_n) = h(t)$$

for every $t \in \mathbb{T}$, by [Lemma 5.5](#).

Similarly, one can prove that

$$\lim_{n \rightarrow \infty} h(t + \alpha_n) = Tu(t)$$

for every $t \in \mathbb{T}$ and conclude that Tu is an almost automorphic function. Thus, Tu is well defined.

Now, let us prove that T is a contraction.

$$\begin{aligned} \|Tz - Ty\| &= \left\| \int_{-\infty}^t X(t)PX^{-1}(\sigma(s))[f(s, z) - f(s, y)]\Delta s \right. \\ &\quad \left. - \int_t^{+\infty} X(t)(I - P)X^{-1}(\sigma(s))[f(s, z) - f(s, y)]\Delta s \right\| \\ &\leq \int_{-\infty}^t Ke_{\Theta_Y}(t, \sigma(s))L\|z - y\|\Delta s + \int_t^{+\infty} Ke_{\Theta_Y}(\sigma(s), t)L\|z - y\|\Delta s \\ &\leq \frac{1}{|\Theta_Y|} [Ke_{\Theta_Y}(t, t) - Ke_{\Theta_Y}(t, -\infty)]L\|z - y\|_{\infty} + \int_t^{+\infty} Ke_Y(t, \sigma(s))L\|z - y\|\Delta s \\ &\leq \frac{1}{|\Theta_Y|} [K - Ke_{\Theta_Y}(t, -\infty)]L\|z - y\|_{\infty} + \frac{1}{\gamma} [K - Ke_Y(t, +\infty)]L\|z - y\|_{\infty}, \end{aligned}$$

by [Theorem 2.13](#). Therefore, we obtain

$$\begin{aligned} \|Tz - Ty\| &\leq \frac{1}{\frac{\gamma}{1+\tilde{\mu}\gamma}} [K - Ke_{\Theta_Y}(t, -\infty)]L\|z - y\|_{\infty} + \frac{1}{\gamma} [K - Ke_Y(t, +\infty)]L\|z - y\|_{\infty} \\ &\leq \frac{1+\tilde{\mu}\gamma}{\gamma} [|K| + |Ke_{\Theta_Y}(t, -\infty)|]L\|z - y\|_{\infty} \\ &\quad + \frac{1}{\gamma} [K + |Ke_Y(t, +\infty)|]L\|z - y\|_{\infty} \\ &\leq L\|z - y\|_{\infty} \left(\frac{2K(1+\tilde{\mu}\gamma)}{\gamma} + \frac{2K}{\gamma} \right) \\ &= L \left(\frac{2K(2+\tilde{\mu}\gamma)}{\gamma} \right) \|z - y\|_{\infty} < \|z - y\|_{\infty}, \end{aligned}$$

by [Theorem 2.14](#).

It follows that T is a contraction, then by the Banach Fixed-Point Theorem, T has a unique fixed point. By the definition of T and [Definition 6.1](#), we obtain that the system [\(6.1\)](#) has a unique solution which is almost automorphic. Therefore, we have the desired result. \square

Remark 6.4. It is clear that the previous theorem remains valid for linear nabla dynamic equations on time scales. In other words, one can prove analogously that the nabla dynamic equation

$$x^\nabla(t) = A(t)x(t) + f(t, x(t)), \quad (6.5)$$

where $A : \mathbb{T} \rightarrow \mathbb{R}^{n \times n}$ and $f : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, has an almost automorphic solution, under similar conditions to the ones presented in [Theorem 5.6](#).

Choosing $\mathbb{T} = \mathbb{R}$ in [Theorem 6.3](#), we obtain a result for semilinear differential equations. It is the content of the next result.

Corollary 6.5. *Let $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuous and almost automorphic with respect to the first variable and $A : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ be almost automorphic, continuous and nonsingular and the set $\{A^{-1}(t)\}_{t \in \mathbb{R}}$ is bounded. Suppose the equation*

$$\dot{x}(t) = A(t)x(t)$$

admits an exponential dichotomy with positive constants K and γ and the following condition holds:

(i) *There exists a constant $0 < L < \frac{\gamma}{4K}$ such that*

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|,$$

for every $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

Then, the system

$$\dot{x}(t) = A(t)x(t) + f(t, x(t))$$

has a unique solution which is almost automorphic.

Taking $\mathbb{T} = \mathbb{Z}$ in [Theorem 6.3](#), we obtain a result for semilinear difference equations. See the result below.

Corollary 6.6. *Let $f : \mathbb{Z} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be regulated and almost automorphic with respect to the first variable and $A : \mathbb{Z} \rightarrow \mathbb{R}^{n \times n}$ be almost automorphic, regulated and nonsingular and $\{A^{-1}(k)\}_{k \in \mathbb{Z}}$ is bounded. Also, suppose that $(I + A(k))$ is nonsingular and the set $\{(I + A(k))^{-1}\}_{k \in \mathbb{Z}}$ is bounded. Suppose the equation*

$$x(k+1) = A(k)x(k)$$

admits an exponential dichotomy with positive constants K and γ and the following condition holds:

(i) There exists a constant $0 < L < \frac{\gamma}{2K(2+\gamma)}$ such that

$$\|f(k, x) - f(k, y)\| \leq L\|x - y\|,$$

for every $x, y \in \mathbb{R}^n$ and $k \in \mathbb{Z}$.

Then, the system

$$x(t+1) = A(t)x(t) + f(t, x(t))$$

has a unique solution which is almost automorphic.

We can also choose $\mathbb{T} = h\mathbb{Z}$ in Theorem 6.3, then it follows a result for a different type of semilinear difference equations. We do not know any result in this direction.

Corollary 6.7. Let $f : h\mathbb{Z} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be regulated and almost automorphic with respect to the first variable and $A : h\mathbb{Z} \rightarrow \mathbb{R}^{n \times n}$ be almost automorphic, regulated and nonsingular and $\{A^{-1}(k)\}_{k \in h\mathbb{Z}}$ is bounded. Also, suppose that $(I + A(k))$ is nonsingular and the set $\{(I + A(k))^{-1}\}_{k \in h\mathbb{Z}}$ is bounded. Suppose the equation

$$x(k+h) = A(k)x(k)$$

admits an exponential dichotomy with positive constants K and γ and the following condition holds:

(i) There exists a constant $0 < L < \frac{\gamma}{2K(2+h\gamma)}$ such that

$$\|f(k, x) - f(k, y)\| \leq L\|x - y\|,$$

for every $x, y \in \mathbb{R}^n$ and $k \in h\mathbb{Z}$.

Then, the system

$$x(k+h) = A(k)x(k) + f(k, x(k))$$

has a unique solution which is almost automorphic.

Finally, we take $\mathbb{T} = \mathbb{P}_{a,b}$ in Theorem 6.3 and we get an interesting result for a different type of equation. We do not know any result in this direction. See the following result.

Corollary 6.8. Let $f : \mathbb{P}_{a,b} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be almost automorphic with respect to the first variable and, for every $k \in \mathbb{Z}$, be continuous at $[k(a+b), k(a+b)+a)$ and regulated at $k(a+b)+a$ and $A : \mathbb{P}_{a,b} \rightarrow \mathbb{R}^{n \times n}$ be almost automorphic, and, for every $k \in \mathbb{Z}$, be continuous at $[k(a+b), k(a+b)+a)$, regulated at $k(a+b)+a$ and nonsingular and $\{A^{-1}(t)\}_{t \in \mathbb{P}_{a,b}}$ is bounded. Also, suppose that $(I + bA(t))$ is nonsingular and $\{(I + bA(t))^{-1}\}_{t \in \mathbb{P}_{a,b}}$ is bounded. Suppose the equation

$$x^\Delta = A(t)x(t)$$

admits an exponential dichotomy with positive constants K and γ and the following condition holds:

(i) There exists a constant $0 < L < \frac{\gamma}{2K(2+b\gamma)}$ such that

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|,$$

for every $x, y \in \mathbb{R}^n$ and $t \in \mathbb{P}_{a,b}$.

Then, the system

$$x^\Delta = A(t)x(t) + f(t, x(t))$$

has a unique solution which is almost automorphic.

7. Examples and applications

In this section, we present some examples and applications of our main results.

Example 7.1. The following economic model is known as a *Keynesian-Cross model with lagged income*. It can be found in [48].

Consider these three equations in a simple closed economy:

$$D(t) = C(t) + I + G; \quad (7.1)$$

$$C(t) = C_0 + cy(t); \quad (7.2)$$

$$y^\Delta(t) = \delta[D^\sigma - y], \quad t \geq a \quad (7.3)$$

where D is the aggregate demand, y is the aggregate income, C is the aggregate consumption, I is the aggregate investment, G is the government spending, $\delta < 1$ is a positive constant known as the *speed of adjustment term* and C_0, c are non-negative constants.

As in [48], we assume that G and I are constants in (7.1), and current consumption is assumed to depend on current income in (7.2). Also, Eq. (7.3) means that the change in income is a fraction of excess demand at $\sigma(t)$ over income at t (see [48]).

Putting (7.1) and (7.2) into (7.3), we obtain

$$y^\Delta = \delta[C_0 + cy^\sigma + I + G - y].$$

Now, using the formula $y^\sigma = y + \mu y^\Delta$ and considering $1 - \delta c\mu(t) \neq 0$ for $t > a$, we have

$$\begin{aligned} y^\Delta &= \frac{\delta(c-1)}{1-\delta c\mu(t)}y + \frac{\delta(c_0 + I + G)}{1-\delta c\mu(t)} \\ &:= f(t)y + g(t). \end{aligned} \quad (7.4)$$

Note that if \mathbb{T} is invariant under translation, then we obtain that the graininess function $\mu(t)$ is almost automorphic, then obviously, by the definition of function g , it follows that g is an almost automorphic function.

Moreover, if we assume that only one of the following inequalities holds, that is, $c < 1$ or $\mu(t) > \frac{1}{c\delta}$ for every $t \in \mathbb{T}$, then we obtain that Eq. (7.4) admits exponential dichotomy and moreover, $\{f(t)^{-1}\}$ is bounded on \mathbb{T} . Assume that $\mu(t) \neq \frac{1}{\delta}$ for every $t \in \mathbb{T}$. In this case, we obtain that the function $f(t)$ is regressive on \mathbb{T} . Indeed, a function $f(t)$ is regressive if $1 + \mu(t)f(t) \neq 0$ for every $t \in \mathbb{T}$. Then,

$$\begin{aligned} 1 + \mu(t)f(t) \neq 0 &\Leftrightarrow 1 + \mu(t) \frac{\delta(c-1)}{1 - \delta c \mu(t)} \neq 0 \\ &\Leftrightarrow 1 - \delta c \mu(t) + \mu(t)\delta(c-1) \neq 0 \Leftrightarrow 1 - \mu(t)\delta \neq 0, \end{aligned}$$

which implies that $\mu(t) \neq \frac{1}{\delta}$. Then, it follows that $f(t)$ is regressive on \mathbb{T} . Therefore, all the hypotheses of Theorem 5.6 are satisfied, and hence we can conclude that Eq. (7.4) has an almost automorphic solution.

The above example generalizes the classical Keynesian-Cross model involving difference equations given in [28]. See [48], for instance.

Now, we present an example which can be found in [52].

Example 7.2. Consider the following nonautonomous dynamic equation

$$x^\Delta(t) = -a(t)x(\sigma(t)) + b(t) \quad (7.5)$$

where $a, b \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$, a, b are almost automorphic functions on \mathbb{T} and $a \in \mathcal{R}$.

It is clear that the equation given by $x^\Delta(t) = -a(t)x(\sigma(t))$ admits exponential dichotomy and also, $\{a(t)^{-1}\}$ is bounded on \mathbb{T} . Notice that the function a takes value in \mathbb{R}_+ and thus, $a(t) \neq 0$, for every $t \in \mathbb{T}$. Thus, taking \mathbb{T} invariant under translations, then all the hypotheses of Theorem 5.6 are satisfied, which implies that Eq. (7.5) has an almost automorphic solution.

We point out that Eq. (7.5) can be used to model many single species models as special cases. For example, taking $\mathbb{T} = \mathbb{R}$ and $x(t) = \frac{1}{N(t)}$, then Eq. (7.5) reduces to the known *Verhulst logistic equation* given by

$$\dot{N}(t) = -N(t)(a(t) - b(t)N(t)).$$

On the other hand, taking $\mathbb{T} = \mathbb{Z}$ and $x(t) = \frac{1}{N(t)}$, then Eq. (7.5) reduces to the known *Beverton–Holt equation* given by

$$N(t+1) = (1 + a(t)) \frac{N(t)}{1 + b(t)N(t)}$$

as explained in [52]. See, for instance, [5] and [47].

Further, if we consider $b(t) = a(t) \ln(c(t))$ and $x(t) = \ln(N(t))$, then when $\mathbb{T} = \mathbb{R}$ Eq. (7.5) reduces to the continuous *Gompertz single species model* given by

$$\dot{N}(t) = a(t)N(t) \ln\left(\frac{c(t)}{N(t)}\right).$$

See [25] and [49], for more details.

Finally, taking $\mathbb{T} = \mathbb{Z}$, we can obtain the discrete *Gompertz single model* which is given by

$$N(t+1) = N(t)^{\frac{1}{1+\alpha(t)}} c(t)^{\frac{\alpha(t)}{1+\alpha(t)}}.$$

See, for instance, [50].

The next example is inspired by Example 4.20 from [36].

Example 7.3. Consider the following equation

$$x^\Delta(t) = Ax(t) + f(t), \quad (7.6)$$

where

$$A = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}, \quad \mu(t) \neq \frac{1}{4} \quad \text{and} \quad f(t) = \begin{pmatrix} \cos\left(\frac{1}{2+\sin t + \sin \sqrt{2}t}\right) \\ \sin \sqrt{2}t + \cos t \end{pmatrix}.$$

By the definition, it is clear that $I + \mu(t)A$ is invertible for all $t \in \mathbb{T}$ and thus, A is regressive. Also, notice that A is invertible and $\{A^{-1}\}$ is bounded on \mathbb{T} . Moreover, since \mathbb{T} is invariant under translations, the graininess function μ is bounded and thus, $\{(I + \mu(t)A)^{-1}\}$ is bounded on \mathbb{T} .

The function f is almost automorphic on \mathbb{T} . Then, using the fact that the eigenvalues of the coefficient matrix in (7.6) are $\lambda_1 = \lambda_2 = -4$ and applying Theorem 5.35 (Putzer Algorithm) from [11], we obtain that the P -matrices are given by

$$P_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$P_1 = (A - \lambda_1 I)P_0 = (A + 4I)P_0.$$

Then, using again Theorem 5.35 (Putzer Algorithm) from [11], we obtain

$$\begin{aligned} r_1^\Delta(t) &= -4r_1(t), & r_1(t_0) &= 1, \\ r_2^\Delta(t) &= r_1(t) - 4r_2(t), & r_2(t_0) &= 0. \end{aligned}$$

Now, calculating r_1 , we get

$$r_1(t) = e_{\ominus 4}(t, t_0)$$

and by the variation of constants formula, we have

$$r_2(t) = e_{\ominus 4}(t, t_0) \int_{t_0}^t \frac{1}{1 - 4\mu(s)} \Delta \tau.$$

Finally, applying Theorem 5.35 (Putzer Algorithm) from [11] again, we get

$$e_A(t, t_0) = r_1(t)P_0 + r_2(t)P_1 = e_{\ominus 4}(t, t_0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, for $t > s$, we get

$$\begin{aligned} \|X(t)P_0X^{-1}(s)\| &= \left\| e_{\ominus 4}(t, t_0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e_4(s, t_0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} e_{\ominus 4}(t, t_0) & 0 \\ 0 & e_{\ominus 4}(t, t_0) \end{pmatrix} \begin{pmatrix} e_4(s, t_0) & 0 \\ 0 & e_4(s, t_0) \end{pmatrix} \right\| \\ &\leq \sqrt{2}e_{\ominus 4}(t, s). \end{aligned}$$

Taking $K = 2$ and $\gamma = 4$, we obtain that Eq. (7.6) admits exponential dichotomy and thus, by Theorem 5.6, we have

$$\begin{aligned} x(t) &= \int_{-\infty}^t X(t)P_0X^{-1}(\sigma(s))f(s)\Delta s + \int_t^{\infty} X(t)(I - P_0)X^{-1}(\sigma(s))f(s)\Delta s \\ &= \int_{-\infty}^t X(t)P_0X^{-1}(\sigma(s))f(s)\Delta s \\ &= \int_{-\infty}^t \begin{pmatrix} e_{\ominus 4}(t, \sigma(s)) & 0 \\ 0 & e_{\ominus 4}(t, \sigma(s)) \end{pmatrix} \begin{pmatrix} \cos\left(\frac{1}{2+\sin t + \sin \sqrt{2}t}\right) \\ \sin \sqrt{2}t + \cos t \end{pmatrix} \Delta s. \end{aligned}$$

In the sequel, we present a model which describes high-order Hopfield neural networks on time scales. We borrow some ideas from [37].

Example 7.4. Consider the following high-order Hopfield neural networks on time scales:

$$\begin{aligned} x_i^{\Delta}(t) &= -c_i(t)x_i(t) + \sum_{j=1}^n a_{ij}j(f_j(x_j(t))) \\ &\quad + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t)g_j(x_j(t))g_l(x_l(t)) + I_i(t), \end{aligned} \tag{7.7}$$

for $i = 1, 2, \dots, n$, where n corresponds to the number of units in a neural network, $x_i(t)$ corresponds to the state vector of the i th unit at the time t , $c_i(t)$ represents the rate with which the i th unit will reset its potential to the resting state in isolation when disconnected from the network external inputs, $a_{ij}(t)$ and $b_{ijl}(t)$ are the first- and second-order connection weights of neural network and $I_i(t)$ denotes the external inputs at time t and f_j and g_j are the activation functions of signal transmission.

Next, we present a result which can be found in [37, Lemma 2.15] which will be essential to our purposes.

Lemma 7.5. *Let $c_i(t)$ be an almost periodic function on \mathbb{T} , where $c_i(t) > 0$, $-c_i(t) \in \mathcal{R}$, $\forall t \in \mathbb{T}$ and*

$$\min_{1 \leq i \leq n} \left\{ \inf_{t \in \mathbb{T}} c_i(t) \right\} = \tilde{m} > 0,$$

then the linear system

$$x^\Delta(t) = \text{diag}(-c_1(t), -c_2(t), \dots, -c_n(t))x(t)$$

admits an exponential dichotomy on \mathbb{T} .

Remark 7.6. A carefully examination of the proof of the above result reveals that we can change the hypothesis concerning almost periodicity of c_i by almost automorphicity and obtain the same conclusion. The proof follows similarly for this other case with obvious adaptations.

Now, we assume that the following conditions are satisfied:

- (H1) $c_i, a_{ij}, b_{ijl}, I_i$ are almost automorphic functions, $-c_i \in \mathcal{R}$ and $c_i > 0$ for every $i, j, l = 1, 2, \dots, n$.
- (H2) There exist positive constants $M_j, N_j, j = 1, 2, \dots, n$ such that $|f_j(x)| \leq M_j$ and $|g_j(x)| \leq N_j$ for $j = 1, 2, \dots, n, x \in \mathbb{R}$.
- (H3) Functions $f_j(u), g_j(u), j = 1, 2, \dots, n$ satisfy the Lipschitz condition, that is, there exist constants $L_j, H_j > 0$ such that $|f_j(u_1) - f_j(u_2)| \leq L_j \|u_1 - u_2\|, |g_j(u_1) - g_j(u_2)| \leq H_j \|u_1 - u_2\|, j = 1, 2, \dots, n$.
- (H4)

$$\max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n \overline{a_{ij}} L_j + \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} N_j H_l + \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} N_l H_j / \underline{c_i} \right\} < 1,$$

where

$$\begin{aligned} \underline{c_i} &= \inf_{t \in \mathbb{T}} |c_i(t)|, & \overline{c_i} &= \sup_{t \in \mathbb{T}} |c_i(t)|, \\ \overline{a_{ij}} &= \sup_{t \in \mathbb{T}} |a_{ij}(t)|, & \overline{b_{ijl}} &= \sup_{t \in \mathbb{T}} |b_{ijl}(t)|, & \overline{I_i} &= \sup_{t \in \mathbb{T}} |I_i(t)|. \end{aligned}$$

Then, by hypotheses (H1), (H2), (H3) and (H4) and using [Lemma 7.5](#), we obtain that all hypotheses of [Theorem 6.3](#) are satisfied, then the system (7.7) possesses a unique almost automorphic solution.

In the sequel, we present an example which can be found in [\[52\]](#). The equation of the following example can be known as *continuous or discrete Lasota–Ważewska model without delay* taking $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, respectively. For more details about this model with delays, see [\[18,35,51\]](#).

Example 7.7. Consider the following dynamic equation on time scales:

$$x^\Delta(t) = -rx(\sigma(t)) + \eta(t)e^{-\gamma x(t)} \quad (7.8)$$

where r, γ are all positive conditions and the initial values of (7.8) are also positive.

Suppose that the function η is almost automorphic. Hence, it is bounded. Let us denote $\tilde{\eta} = \sup_{t \in \mathbb{T}} \eta(t)$. Also, define the function $g(t, x(t)) = \eta(t)e^{-\gamma x(t)}$. Then, we have

$$|g(t, x_1(t)) - g(t, x_2(t))| = |\eta(t)e^{-\gamma x_1(t)} - \eta(t)e^{-\gamma x_2(t)}| \leq \tilde{\eta}\gamma |x_1(t) - x_2(t)|.$$

Consider $\tilde{\eta}\gamma < \frac{\gamma_1}{2K(2+\tilde{\mu}\gamma_1)}$, where K, γ_1 are the constants from the exponential dichotomy condition and $\tilde{\mu} = \sup_{t \in \mathbb{T}} |\mu(t)|$. Assume also that \mathbb{T} is invariant under translations, we obtain that all hypotheses of [Theorem 6.3](#) are satisfied which implies that Eq. (7.8) has an almost automorphic solution.

Finally, we present an example which can be found in [\[52\]](#). Notice that for a specific time scale, that is, $\mathbb{T} = \mathbb{R}$, the equation in the following example reduces to a single artificial effective neuron with dissipation model. See, for instance, [\[25\]](#) and [\[31\]](#).

Example 7.8. Consider the dynamic equation given by

$$x^\Delta(t) = -a(t)x(\sigma(t)) + b(t) \tanh(x(t)) + \gamma(t) \quad (7.9)$$

where $a, b, \gamma \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$ and a, b, γ are almost automorphic functions and regressives on \mathbb{T} . It is clear that the equation given by $x^\Delta(t) = -a(t)x(\sigma(t))$ admits exponential dichotomy and also, $\{a(t)^{-1}\}$ is bounded on \mathbb{T} . Notice that the function a takes value in \mathbb{R}_+ and thus, $a(t) \neq 0$, for every $t \in \mathbb{T}$.

Moreover, notice that $|\tanh(x_1) - \tanh(x_2)| \leq |x_1 - x_2|$ for $x_1, x_2 \in \mathbb{T}$. Then, denoting $g(t, x(t)) = b(t) \tanh(x(t)) + \gamma(t)$, we obtain

$$\begin{aligned} |g(t, x_1(t)) - g(t, x_2(t))| &= |b(t) \tanh x_1(t) + \gamma(t) - b(t) \tanh x_2(t) - \gamma(t)| \\ &\leq |b(t)| |x_1(t) - x_2(t)| \leq \tilde{b} |x_1(t) - x_2(t)|, \end{aligned}$$

where $\tilde{b} = \sup_{t \in \mathbb{T}} |b(t)|$. If we suppose that $\tilde{b} < \frac{\gamma_1}{2K(2+\tilde{\mu}\gamma)}$, where γ_1, K are the constants from the exponential dichotomy and $\tilde{\mu} = \sup_{t \in \mathbb{T}} |\mu(t)|$ and assume that \mathbb{T} is invariant under translations, then all hypotheses of [Theorem 6.3](#) are satisfied which implies that Eq. (7.9) has an almost automorphic solution.

References

- [1] M. Adivar, Y.N. Raffoul, Existence results for periodic solutions of integro-dynamic equations on time scales, *Ann. Mat. Pura Appl.* (4) 188 (4) (2009) 543–559.
- [2] D.R. Anderson, J. Hoffacker, Positive periodic time-scale solutions for functional dynamic equations, *Aust. J. Math. Anal. Appl.* 3 (1) (2006), Art. 5, 14 pp.
- [3] D. Araya, R. Castro, C. Lizama, Almost automorphic solutions of difference equations, *Adv. Difference Equ.* (2009), <http://dx.doi.org/10.1155/2009/591380>, Article ID 591380, 15 pp.
- [4] F.M. Atici, D.C. Biles, A. Lebedinsky, An application of time scales to economics, *Math. Comput. Modelling* 43 (2006) 718–726.
- [5] R.J. Beverton, S.J. Holt, The theory of fishing, in: M. Graham (Ed.), *Sea Fisheries; Their Investigation in the United Kingdom*, Edward Arnold, London, 1956, pp. 372–441.
- [6] L. Bi, M. Bohner, M. Fan, Periodic solutions of functional dynamic equations with infinite delay, *Nonlinear Anal.* 68 (7) (2008) 170–174.
- [7] S. Bochner, Uniform convergence of monotone sequences of functions, *Proc. Natl. Acad. Sci. USA* 47 (1961) 582–585.
- [8] S. Bochner, A new approach in almost-periodicity, *Proc. Natl. Acad. Sci. USA* 48 (1962) 2039–2043.
- [9] S. Bochner, Continuous mappings of almost automorphic and almost periodic functions, *Proc. Natl. Acad. Sci. USA* 52 (1964) 907–910.
- [10] M. Bohner, M. Fan, J. Zhang, Periodicity of scalar dynamic equations and applications to population models, *J. Math. Anal. Appl.* 300 (1) (2007) 1–9.
- [11] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.
- [12] M. Bohner, A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.
- [13] D. Brigo, F. Mercurio, Discrete time vs continuous time stock-price dynamics and implications for option pricing, *Finance Stoch.* 4 (2000) 147–159.
- [14] T. Caraballo, D. Cheban, Almost periodic and almost automorphic solutions of linear differential/difference equations without Favard’s separation condition. I, *J. Diff. Equations* 246 (1) (2009) 108–128.
- [15] T. Caraballo, D. Cheban, Almost periodic and almost automorphic solutions of linear differential/difference equations without Favard’s separation condition. II, *J. Diff. Equations* 246 (3) (2009) 1164–1186.
- [16] T. Caraballo, D. Cheban, Levitan/Bohr almost periodic and almost automorphic solutions of second order monotone differential equations, *J. Diff. Equations* 251 (3) (2011) 708–727.
- [17] Z. Chen, W. Lin, Square-mean pseudo almost automorphic process and its application to stochastic evolution equations, *J. Funct. Anal.* 261 (1) (2011) 69–89.
- [18] S.N. Chow, Existence of periodic solutions of autonomous functional differential equations, *J. Diff. Equations* 15 (1974) 350–378.
- [19] F.B. Christiansen, T.M. Fenchel, *Theories of Populations in Biological Communities*, Lectures Notes in Ecological Studies, vol. 20, Springer-Verlag, Berlin, 1977.
- [20] P. Cieutat, K. Ezzinbi, Almost automorphic solutions for some evolution equations through the minimizing for some subvariant functional, applications to heat and wave equations with nonlinearities, *J. Funct. Anal.* 260 (9) (2011) 2598–2634.
- [21] C. Corduneanu, *Almost Periodic Functions*, John Wiley and Sons, New York, 1968.
- [22] A. Dogan, J. Graef, L. Kong, Higher-order singular multi-point boundary-value problems on time scales, *Proc. Edinb. Math. Soc.* (2) 54 (2) (2011) 345–361.
- [23] L. Erbe, J. Baoguo, A. Peterson, Belohorec-type oscillation theorem for second order sublinear dynamic equations on time scales, *Math. Nachr.* 284 (13) (2011) 1658–1668.
- [24] L. Erbe, J. Baoguo, A. Peterson, On the asymptotic behavior of solutions of Emden–Fowler equations on time scales, *Ann. Mat. Pura Appl.* (4) 191 (2) (2012) 205–217.
- [25] M. Fan, D. Ye, Convergence dynamics and pseudo almost periodicity of a class of nonautonomous RFDEs with applications, *J. Math. Anal. Appl.* 309 (2005) 598–625.
- [26] M. Federson, J.G. Mesquita, A. Slavík, Measure functional differential equations and functional dynamic equations on time scales, *J. Diff. Equations* 252 (2012) 3816–3847.
- [27] M. Federson, J.G. Mesquita, A. Slavík, Basic results for functional differential and dynamic equations involving impulses, *Math. Nachr.* 286 (2–3) (2013) 181–204.
- [28] B.S. Ferguson, G.C. Lim, *Dynamic Economic Models in Discrete Time: Theory and Empirical Applications*, Routledge, London, 2003.

- [29] M. Fu, Z. Liu, Square-mean almost automorphic solutions for some stochastic differential equations, *Proc. Amer. Math. Soc.* 138 (10) (2010) 3689–3701.
- [30] J.A. Goldstein, G.M. N'Guérékata, Almost automorphic solutions of semilinear evolution equations, *Proc. Amer. Math. Soc.* 133 (8) (2005) 2401–2408.
- [31] A. Gopalsamy, S. Saryasa, Time delays and stimulus dependent pattern formation in periodic environments in isolated neurons, *Dyn. Contin. Discrete Impuls. Syst. B.* 9 (2002) 39–58.
- [32] S. Hilger, Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten, PhD thesis, Universität Würzburg, 1988.
- [33] S. Keller, Asymptotisches Verhalten invarianter Faserbündel bei Diskretisierung und Mittelwertbildung in Rahmen der Analysis auf Zeitskalen, PhD thesis, Universität Augsburg, 1999.
- [34] I. Klapper, H. Qian, Remarks on discrete and continuous large-scale models of DNA dynamics, *Biophys. J.* 74 (1998) 2504–2514.
- [35] W.T. Li, S.S. Cheng, Asymptotic properties of the positive equilibrium of a discrete survival model, *Appl. Math. Comput.* 157 (2004) 29–38.
- [36] Y. Li, C. Wang, Uniformly almost periodic functions and almost periodic solutions to dynamic equations on time scales, *Abstract Appl. Anal.* 2011 (2011), 22 pp.
- [37] Y. Li, C. Wang, Almost periodic functions on time scales and applications, *Discrete Dyn. Nat. Soc.* 2011 (2011), 20 pp.
- [38] Y. Li, C. Wang, Pseudo almost periodic functions and pseudo almost periodic solutions to dynamic equations on time scales, *Adv. Difference Equ.* 2012 (2012) 77, 24 pp.
- [39] Xi-Lan Liu, Wan-Tong Li, Periodic solutions for dynamic equations on time scales, *Nonlinear Anal.* 67 (5) (2007) 1457–1463.
- [40] J. Liu, X. Song, Almost automorphic and weighted pseudo almost automorphic solutions of semilinear evolution equations, *J. Funct. Anal.* 258 (1) (2010) 196–207.
- [41] C. Lizama, J.G. Mesquita, Almost automorphic solutions of nonautonomous difference equations, *J. Math. Anal. Appl.* (2013), in press.
- [42] G.M. N'Guérékata, *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*, Kluwer Academic/Plenum Publishers, New York, 2001.
- [43] A. Slavík, Product integration on time scales, *Dynamic Syst. Appl.* 19 (2010) 97–112.
- [44] A. Slavík, Averaging dynamic equations on time scales, *J. Math. Anal. Appl.* 388 (2012) 996–1012.
- [45] A. Slavík, Dynamic equations on time scales and generalized ordinary differential equations, *J. Math. Anal. Appl.* 385 (2012) 534–550.
- [46] H.R. Sun, W.T. Li, Existence theory for positive solutions to one-dimensional p -Laplacian boundary value problems on time scales, *J. Diff. Equations* 240 (2) (2007) 217–248.
- [47] S. Tang, R.A. Cheke, Y. Xiao, Optimal impulsive harvesting on non-autonomous Beverton–Holt difference equations, *Nonlinear Anal.* 65 (2006) 2311–2341.
- [48] C.C. Tisdell, A. Zaidi, Basic qualitative and quantitative results for solutions to nonlinear, dynamic equations on time scales with an application to economic modelling, *Nonlinear Anal.* 68 (11) (2008) 3504–3524.
- [49] A. Tsoularis, J. Wallace, Analysis of logistic growth models, *Math. Biosci.* 179 (2002) 21–55.
- [50] P. Turchin, *Complex Population Dynamics: A Theoretical/Empirical Synthesis*, Princeton University Press, 2003.
- [51] M. Wazewska-Czyżewskam, A. Lasota, Mathematical problems of the dynamics of a system of red blood cells, *Ann. Polish Math. Soc. Ser. III, Appl. Math.* 17 (1976) 23–40.
- [52] J. Zhang, M. Fan, Boundedness and stability of semi-linear dynamic equations on time scales, preprint, 2012.