The scattering problem for Hamiltonian ABCD Boussinesq systems in the energy space

Chulkwang Kwak [a,1], Claudio Muñoz [b,*,2], Felipe Poblete [c,3], Juan C. Pozo [d,4]

[a] *Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Campus San Joaquín. Avda. Vicuña Mackenna 4860, Santiago, Chile*
[b] *CNRS and Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático (UMI 2807 CNRS), Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile*
[c] *Universidad Austral de Chile, Facultad de Ciencias, Instituto de Ciencias Físicas y Matemáticas, Valdivia, Chile*
[d] *Departamento de Matemática y Estadística, Facultad de Ciencias, Universidad de La Frontera, Casilla 54-D, Temuco, Chile*

A R T I C L E   I N F O

A B S T R A C T

The Boussinesq *abcd* system is a 4-parameter set of equations posed in $\mathbb{R}_t \times \mathbb{R}_x$, originally derived by Bona, Chen and Saut [11,12] as first order 2-wave approximations of the incompressible and irrotational, two dimensional water wave equations in the shallow water wave regime, in the spirit of the original Boussinesq derivation [17]. Among many particular regimes, depending each of them in terms of the value of the parameters (a, b, c, d) present in the equations, the *generic* regime is characterized by the setting $b, d > 0$ and $a, c < 0$. If additionally $b = d$, the *abcd* system is Hamiltonian. The equations in this regime are globally well-posed in the energy space $H^1 \times H^1$, provided one works with small solutions [12]. In this paper, we investigate decay and the scattering problem in this regime, which is characterized as having (quadratic) long-range nonlinearities, very weak linear decay $O(t^{-1/3})$ because of the one dimensional setting, and existence of non scattering solutions (solitary waves). We prove, among other results, that for a sufficiently *dispersive abcd* systems (characterized only in terms of parameters a, b and c), all small solutions must decay to zero, locally strongly in the energy space, in proper subset of the light cone $|x| \leq |t|$. We prove this result by constructing three suitable virial functionals in the spirit of works [27,28], and more precisely [42] (valid for the simpler scalar "good Boussinesq" model), leading to global in time decay and control of all local $H^1 \times H^1$ terms. No parity nor extra decay assumptions are needed to prove decay, only small solutions in the energy space.

---

* Corresponding author.
 E-mail addresses: chkwak@mat.uc.cl (C. Kwak), cmunoz@dim.uchile.cl (C. Muñoz), felipe.poblete@uach.cl (F. Poblete), juan.pozo@ufrontera.cl (J.C. Pozo).

R É S U M É

Le système de Boussinesq *abcd* est une famille à 4 paramètres d'équations posées en
$\mathbb{R}_t \times \mathbb{R}_x$, derivées par Bona, Chen et Saut [11,12], comme une approximation 2-onde
des èquations de type *water waves* en 2d, en améliorant la dérivation originale de
Boussinesq [17]. Entre les valeurs (a, b, c, d) admis, le régime générique est donné
par les choix $b, d > 0$ et $a, c < 0$. Si en addition $b = d$, alors le système *abcd*
est Hamiltonien. Les équations dans ce régime sont globalement bien poseés dans
l'espace d'énergie $H^1 \times H^1$, pour de données petites [12]. Dans ce papier, on
étude la décroissance des solutions pour ce régime, caractérisé par de nonlinearitées
quadratiques, décroissance linéaire $O(t^{-1/3})$, et l'existence d'ondes solitaires. On
montre que, dans un certain régime de paramètres (a, b, c, d), chaque solution petite
converge vers zéro, localement en espace. On prouve ce résultat avec l'aide de trois
fonctionnelles du type Virial, comme dans [27,28,42].

© 2018 Elsevier Masson SAS. All rights reserved.

1. Introduction and main results

1.1. Setting of the problem

This paper concerns the study of nonlinear scattering and decay properties for the one-dimensional *abcd*
system introduced by Bona, Chen and Saut [11,12]:

$$\begin{cases} (1 - b\,\partial_x^2)\partial_t\eta + \partial_x\big(a\,\partial_x^2 u + u + u\eta\big) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ (1 - d\,\partial_x^2)\partial_t u + \partial_x\big(c\,\partial_x^2\eta + \eta + \tfrac{1}{2}u^2\big) = 0. \end{cases} \tag{1.1}$$

Here $u = u(t, x)$ and $\eta = \eta(t, x)$ are real-valued scalar functions. This equation was originally derived by
Bona, Chen and Saut [11] as a first order, one dimensional asymptotic regime model of the water waves
equation, in the vein of the Boussinesq original derivation [17], but maintaining all possible equivalences
between the involved physical variables, and taking into account the shallow water regime. The physical
perturbation parameters under which the expansion is performed are

$$\alpha := \frac{A}{h} \ll 1, \quad \beta := \frac{h^2}{\ell^2} \ll 1, \quad \alpha \sim \beta,$$

and where A and ℓ are typical waves amplitude and wavelength respectively, and h is the constant depth.
Equations (1.1) are part of a hierarchy of Boussinesq models, including second order systems, which were
obtained in [11] and [13].

A rigorous justification for this model (1.1) from the free surface Euler equations (as well as extensions
to higher dimensions) was given by Bona, Colin and Lannes [13], see also Alvarez-Samaniego and Lannes [7]
for improved results. Since then, these models have been extensively studied in the literature, see e.g. [14,
13,33,45] and references therein for a detailed account of known results. This list is by no means exhaustive.

The constants in (1.1) are not arbitrary and follow the conditions [11]

$$a + b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \quad c + d = \frac{1}{2}(1 - \theta^2) \geq 0, \tag{1.2}$$

for some $\theta \in [0, 1]$. Moreover, $a + b + c + d = \frac{1}{3}$ is independent of θ. (This case is referred as the regime
without surface tension $\tau \geq 0$, otherwise one has parameters (a, b, c, d) such that $a + b + c + d = \frac{1}{3} - \tau$.)

In this paper, we will concentrate in the *Hamiltonian generic* case [12, p. 932], namely the case where

$$b, d > 0, \quad a, c < 0, \quad b = d. \tag{1.3}$$

In the case (1.3), it is well-known [12] that (1.1) is globally well-posed in $H^s \times H^s$, $s \geq 1$, for small data, thanks to the preservation of the energy

$$E[u, \eta](t) := \frac{1}{2} \int \left(-a(\partial_x u)^2 - c(\partial_x \eta)^2 + u^2 + \eta^2 + u^2 \eta \right)(t, x) dx, \tag{1.4}$$

which is conserved by the flow. Since now, we will identify $H^1 \times H^1$ as the standard *energy space* for (1.1).

As for the low regularity Cauchy problem associated to (1.1) and its generalizations to higher dimensions, Saut et al. [46,47] studied in great detail the long time existence problem by focusing in the small physical parameter ε appearing from the asymptotic expansions. They showed well-posedness (on a time interval of order $1/\varepsilon$) for (1.1). Previous results by Schonbek [48] and Amick [8] considered the case $a = c = d = 0$, $b = \frac{1}{3}$, a version of the original Boussinesq system, proving global well-posedness under a non cavitation condition, and via parabolic regularization. Linares, Pilod and Saut [33] considered existence in higher dimensions for time of order $O(\varepsilon^{-1/2})$, in the KdV–KdV regime ($b = d = 0$). Additional low-regularity well-posedness results can be found in the recent work by Burtea [18]. On the other hand, ill-posedness results and blow-up criteria (for the case $b = 1$, $a = c = 0$), are proved in [21], following the ideas in Bona–Tzvetkov [16].

By considering the new stretching of variables $u(t/\sqrt{b}, x/\sqrt{b})$ and $\eta(t/\sqrt{b}, x/\sqrt{b})$, we can assume $b = d = 1$ and rewrite (1.1) as the slightly simplified model

$$\begin{cases} (1 - \partial_x^2)\partial_t \eta + \partial_x \left(a\,\partial_x^2 u + u + u\eta \right) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ (1 - \partial_x^2)\partial_t u + \partial_x \left(c\,\partial_x^2 \eta + \eta + \frac{1}{2}u^2 \right) = 0, \end{cases} \tag{1.5}$$

with new conditions (see (1.2)–(1.3))

$$a, c < 0, \quad a + 1 = \frac{1}{2b}\left(\theta^2 - \frac{1}{3} \right), \quad c + 1 = \frac{1}{2b}(1 - \theta^2) \geq 0, \tag{1.6}$$

for some $\theta \in [0, 1]$. Note that in particular now $c > -1$, and equation (1.5) will be the main subject of this paper.

Since (1.5) under (1.6) is globally well-posed in the energy space, at least for small solutions, knowing the asymptotic behavior of solutions for large time is of important relevance. However, being (1.5) a one-dimensional model, decay and scattering properties are often difficult to prove, in particular for nonlinearities (such as the ones present in (1.5) which are long range of not too flat at the origin, with respect to the expected scattering dynamics. Moreover, sometimes not even decay is allowed because of the existence of solitary waves which do not decay in the energy space. Therefore, nonlinear decay and scattering estimates must take into account these obstructions to decay.

1.2. Decay via virial functionals

In a series of papers [27,28], the above problematics were attacked by using techniques directly related to the dynamics of dispersive solutions in the energy space. Decay properties of one dimensional wave models in the energy space were proved in presence of very mild assumptions and strong enemies, such as variable coefficients, presence of internal modes, and quadratic nonlinearities. The case studied in [28], the one dimensional Klein–Gordon equation

$$\partial_t^2 u - \partial_x^2 u + mu + f(u) = 0, \quad m \in \mathbb{R},$$

although in part a consequence of the more involved results in [27] (see also [49]), is specially relevant to the contents of this paper, since it considers precisely perturbations of the vacuum solution, which is assumed to be stable for all time. In particular, the following result was proved:

Theorem 1.1 ([28]). *Assume that f is odd and $|f'(u)| \leq C|u|^{p-1}$, $|u| < 1$. Then any globally defined small, odd solution in the energy space must locally decay to zero in space: for any sufficiently small ε,*

$$\sup_{t \in \mathbb{R}} \|(u, u_t)(t)\|_{H^1 \times L^2} < \varepsilon \implies \lim_{t \to \pm\infty} \|(u, u_t)(t)\|_{(H^1 \times L^2)(I)} = 0, \tag{1.7}$$

for any compact interval I.

Remark 1.1. This result can be improved [29] by assuming f not necessarily odd, but the flatness condition

$$\exists C > 0 \quad \text{s.t.} \quad \frac{1}{2}uf(u) - F(u) \leq C|u|^6, \quad \forall u \in \mathbb{R}, \quad F(u) := \int_0^u f(s)ds,$$

is needed. This condition is satisfied for instance if f is defocusing, or f has focusing behavior, but L^2-supercritical at the origin.

The essential point in the proof of (1.7) was the following *virial identity*: for any smooth, bounded $\psi = \psi(x)$, one has

$$-\frac{d}{dt} \int u_t \left(\psi u_x + \frac{1}{2}\psi' u \right) = \int \psi' u_x^2 - \frac{1}{4} \int \psi''' u^2 - \int \psi' \left(\frac{1}{2}uf(u) - F(u) \right),$$

which reveals that local control on (u, u_x) can be obtained by proving the coercivity estimate

$$\int \psi' u_x^2 - \frac{1}{4} \int \psi''' u^2 - \int \psi' \left(\frac{1}{2}uf(u) - F(u) \right) \gtrsim \int \psi'(u^2 + u_x^2).$$

Precisely, this lower bound was obtained under the smallness and oddness assumptions stated in Theorem 1.1. Roughly speaking, proving

$$-\frac{d}{dt} \int u_t \left(\psi u_x + \frac{1}{2}\psi' u \right) \gtrsim \int \psi'(u^2 + u_x^2),$$

implies that the local H^1 norm of u is converging to zero for large time, at least along a sequence of times. (See also [37–40] for preliminary introduction of virial identities to the study of the blow up dynamics in critical gKdV and NLS equations.) The reader may also consider the work by Lindblad and Tao [34], where similar averaged decay estimates were obtained.

In some sense, Theorem 1.1 can be recast as proving convergence to zero in time of the local energy norm, by-passing the standard linear decay estimates, that is to say, without proving convergence to a (modified or not) linear profile, which is well-known that decays.

The versatility of the previous result, Theorem 1.1, can be measured by the following two completely unrelated problems, for which very similar conclusions can be obtained, and even improving the conclusion in (1.7), in the sense that the interval I in (1.7) can be chosen *growing as time evolves*. First, consider the quasilinear $1+1$ model known as the Born–Infeld model

$$(1 + (\partial_x u)^2)\partial_t^2 u + 2\partial_t u \, \partial_x u \, \partial_{tx}^2 u - (1 + (\partial_t u)^2)\partial_x^2 u = 0, \quad u(t, x) \in \mathbb{R}. \tag{1.8}$$

Let $C_0 > 0$ be any large but fixed constant. In what follows, we will need the following, time-dependent, space interval:

$$I(t) := \left(-\frac{C_0|t|}{\log^2 |t|}, \frac{C_0|t|}{\log^2 |t|}\right), \quad |t| \geq 2. \tag{1.9}$$

Note that this interval is a slightly proper subset of the standard light cone $\{|x| \leq |t|\}$, associated to wave-like equations. For equation (1.8), the following result was proved:

Theorem 1.2 ([4]). *There exists an $\varepsilon > 0$ such that if $(u, \partial_t u)$ solves (1.8) with*

$$\sup_{t \in \mathbb{R}} \|(u, \partial_t u)(t)\|_{H^s \times H^{s-1}} < \varepsilon, \quad s > \frac{3}{2}, \tag{1.10}$$

then one has, for any $C > 0$ arbitrarily large and $I(t)$ defined in (1.9),

$$\lim_{t \to \pm\infty} \|(u, \partial_t u)(t)\|_{(H^1 \times L^2)(I(t))} = 0. \tag{1.11}$$

Note that condition (1.10) is needed in order to ensure local well-posedness for the quasilinear model (1.8). Also, it is needed for a good definition of virial identities, see [4] for more details. Since $u(t, x) := \phi(x \pm t)$ is solution of (1.8), for any ϕ smooth, (1.11) establishes almost sharp decay estimates for the 1+1 dimensional BI model.

The second problem for which interesting conclusions can be stated is a classic one-wave fluid model. In [42], the authors considered extending the previous results to the case of the *good Boussinesq model* in $1+1$ dimensions,

$$\begin{aligned}
\partial_t^2 u - \partial_x^2 u - \partial_x^4 u + \partial_x^2 f(u) &= 0, \quad u(t, x) \in \mathbb{R}, \\
f(0) &= 0, \quad |f'(s)| \lesssim |s|^{p-1}, \, |s| < 1.
\end{aligned} \tag{1.12}$$

This equation represents the simplest regularization of the originally derived (ill-posed), one-wave Boussinesq equation, which contains the fourth order term with positive sign: $+\partial_x^4 u$. It is also a canonical model of shallow water waves as well as the Korteweg–de Vries (KdV) equation, see e.g. [15]. Bona and Sachs [15], Linares [31], and Liu [35,36], established that (1.5) is locally well-posed (and even globally well-posed for small data [15,31]) in the standard energy space for $(u, \partial_t \partial_x^{-1} u) \in H^1 \times L^2$. For this problem the following decay estimate was proved:

Theorem 1.3 ([42]). *Consider (1.12) in system form:*

$$\begin{cases}
\partial_t u_1 = \partial_x u_2, \quad u_1 := u, \\
\partial_t u_2 = \partial_x(u_1 - \partial_x^2 u_1 - f(u_1)).
\end{cases} \tag{1.13}$$

There exists an $\varepsilon > 0$ such that if

$$\|(u_1, u_2)(t = 0)\|_{H^1 \times L^2} < \varepsilon,$$

then one has, for any $C > 0$ arbitrarily large and $I(t)$ defined in (1.9),

$$\lim_{t \to \pm\infty} \|(u_1, u_2)(t)\|_{(H^1 \times L^2)(I(t))} = 0. \tag{1.14}$$

Note that this decay result holds even in the energy space, and even in the presence of solitary waves. In some sense, it can be referred as a *nonlinear scattering* result, because even nonlinear waves are considered as possible cases in (1.14). Previous results on scattering of small amplitude solutions of (1.13) were proved by Liu [35], Linares–Sialom [32], and Cho–Ozawa [22]. In these works, decay was proved in weighted spaces, and for $p \geq p_c$, a certain critical exponent for (subcritical or critical modified) scattering.

1.3. Main results

In this paper we improve the results obtained in [42] for the scalar good Boussinesq model, by considering now the physically more representative, and mathematically more involved *abcd* system of Boussinesq equations (1.5), under (1.6). We prove, using well-chosen virial functionals, that all small globally defined $H^1 \times H^1$ solutions must decay in time in time-growing intervals in space, slightly proper subsets of the light cone. In order to rigorously state such a result, we need first a quantitative notion of dispersion in terms of the parameters present in the equations.

Definition 1.1 *(Dispersion-like parameters).* We say that (a, b, c) satisfying (1.6) are *dispersion-like parameters* if additionally

$$3(a + c) + 2 < 8ac. \tag{1.15}$$

Condition (1.15) formally says that (a, c), being both negative, need to be sufficiently far from zero, depending on the remaining parameter $b > 0$. Indeed, for values of a and c sufficiently close to zero, (1.15) is not satisfied. From the mathematical point of view, this condition can be understood as having *sufficient dispersion* in (1.5) to allow decay to zero for both components of the flow.

Remark 1.2. We also believe that some of the conclusions of this paper are still valid if Condition (1.15) is replaced by a weaker assumption, following the new ideas introduced in [30], but with substantially more difficult proofs.

Additionally, along this paper we will prove the following sufficient conditions:

Lemma 1.1 *(Sufficient conditions for existence of dispersion-like parameters). Assume $a, c < 0$. Then the following holds:*

(i) *The remaining conditions in (1.6) hold for a continuous segment of parameters (a, c), if and only if $b > \frac{1}{6} \sim 0.17$ (cf. Fig. 1). In other words, the abcd Boussinesq system in the regime $a, c < 0$ and $b = d > 0$ makes sense only for $b > \frac{1}{6}$.*

(ii) *Moreover, if in addition $b > \frac{2}{9} \sim 0.22$, then the dispersion-like condition (1.15) holds for a continuous segment of nonempty parameters (a, c), described in Fig. 4.*

Since condition (1.15) is nonempty or b "large enough", we are ready to state the main result of this paper.

Theorem 1.4. *Let $(u, \eta) \in C(\mathbb{R}, H^1 \times H^1)$ be a global, small solution of (1.5)–(1.6), such that for some $\varepsilon > 0$ small*

$$\|(u, \eta)(t = 0)\|_{H^1 \times H^1} < \varepsilon. \tag{1.16}$$

Assume additionally that (a, c) are dispersion-like parameters as in (1.15). *Then, for $I(t)$ as in* (1.9), *there is strong decay:*

$$\lim_{t \to \pm\infty} \|(u, \eta)(t)\|_{(H^1 \times H^1)(I(t))} = 0. \tag{1.17}$$

Some remarks are important to mention.

Remark 1.3. From (1.16) and (1.4) one readily has $\sup_{t \in \mathbb{R}} \|(u, \eta)(t)\|_{H^1 \times H^1} \lesssim \varepsilon$, that is, zero is stable for $a, c < 0$.

Remark 1.4. Theorem 1.4 is, as far as we know, the first decay-in-time result for the original, one dimensional *abcd* Boussinesq systems, which contains some of the most challenging ingredients in scattering theory. See also Muñoz and Rivas [43], where decay of order $O(t^{-1/3})$ is proved in weighted Sobolev spaces, for the case of a generalized *abcd* system with $a = c$, $b = d$ and nonlinearities $\partial_x(u^p \eta)$ and $\partial_x(u^{p+1}/(p+1))$, and power $p \geq 5$. However, equations (1.5) (characterized by the exponent $p = 1$) involve three important challenging issues: *long range* (quadratic) nonlinearities, very weak linear dispersion $O(t^{-1/3})$ inherent to the one dimensional setting, and the existence of (non scattering) solitary waves, standard enemies to decay.

Remark 1.5 *(About the rate of decay).* Theorem 1.4 provides a mild rate of decay for solutions in the energy space: it turns out that (see (5.1))

$$\int\limits_2^\infty \int e^{-c_0|x|} \left(u^2 + \eta^2 + (\partial_x u)^2 + (\partial_x \eta)^2 \right)(t, x) dx \, dt \lesssim_{c_0} \varepsilon^2, \tag{1.18}$$

which reveals that local-in-space H^1 norms do integrate in time. This decay estimate may be compared with the formal rate of decay of the linear dynamics, that should be in general $O_{L^\infty}(t^{-1/3})$ [43] (recall that not all a, b, c lead to linear systems with the same rate of decay). In that sense, (1.18) reveals that a hidden improved decay mechanism is present in (1.5). No better estimate can be obtained from our methods, as far as we understand, unless additional assumptions (on the decay of the initial data) are a priori granted. Additionally, let us emphasize that the linear rate of decay leads (at least formally) to the emergence of modified scattering (or "supercritical scattering" because of long range nonlinearities). See [27,29] for a complete discussion on this topic.

Remark 1.6. The interval $I(t)$ can be made slightly more precise, for instance,

$$\tilde{I}(t) := \left(-\frac{C_0|t|}{\log|t| \log^2(\log|t|)}, \frac{C_0|t|}{\log|t| \log^2(\log|t|)} \right), \quad |t| \geq 11,$$

or subsequent $\log(\cdots)$ improvements, are also valid. However, the interval with ends $\pm \frac{C_0|t|}{\log|t|}$ cannot be reached by our results.

Remark 1.7. Theorem 1.4 is also in formal agreement with the expected group velocity of (1.5) linear waves. It is well-known that the dispersion relation is given by

$$w(k) = \frac{\pm|k|}{1 + k^2} (1 - ak^2)^{1/2} (1 - ck^2)^{1/2},$$

which implies the following group velocity

$$|w'(k)| = \frac{|ack^6 + 3ack^4 - (1 + 2a + 2c)k^2 + 1|}{(1 + k^2)^2 (1 - ak^2)^{1/2} (1 - ck^2)^{1/2}}.$$

It can be proved (see more details in Appendix A), that $|w'(k)| > 0$ for all $k \in \mathbb{R}$ if $b \geq \frac{2}{9}$. Compare also with the conclusions in Corollary 1.1 on solitary waves and their speeds.

Remark 1.8. Based on some formal computations, we believe that condition (1.15) is not sharp (but close to sharp), and Theorem 1.4 should hold for values of b below $\frac{2}{9}$, but with harder proofs.

Theorem 1.4 in some sense corroborates the versatility of the Virial technique when proving decay, bypassing the proof of existence of an asymptotic (modified) linear profile by standard scattering techniques. These last methods need to exclude the existence of small solitary waves by using weighted norms, an assumption that is not necessary in our methods. Precisely, Theorem 1.4 is in concordance with the existence of solitary waves for (1.5).

The proof of Theorem 1.4, as all previously stated results, is based in the introduction of suitable virial identities from which local dispersion in the energy space is clearly identified. Since (1.5) is a two-component system, any virial term must contain contributions from both variables u and η. It turns out that the main virial term is given by

$$\mathcal{I}(t) := \int \varphi(u\eta + \partial_x u \partial_x \eta),$$

where $\varphi = \varphi(x)$ is a smooth and bounded function. The introduction of the functional $\mathcal{I}$ is simple to explain, and it is directly inspired by the momentum

$$P[u, \eta](t) := \int (u\eta + \partial_x u \partial_x \eta)(t, x) dx,$$

which is conserved by the $H^1 \times H^1$ flow [12]. However, in order to prove Theorem 1.4, we will need to introduce two new functionals:

$$\mathcal{J}(t) := \int \varphi' \eta \partial_x u, \quad \text{and} \quad \mathcal{K}(t) := \int \varphi' \partial_x \eta u.$$

The functionals $\mathcal{J}$ and $\mathcal{K}$ can be understood as second order corrections to the main functional $\mathcal{I}(t)$, and allow us to cancel out several bad terms appearing from the variation in time of $\mathcal{I}(t)$. It is important to notice that $\mathcal{J}(t) + \mathcal{K}(t)$ is somehow technically useless because it contains the total derivative $\partial_x(u\eta)$, so as far as we need it, we will modify $\mathcal{I}(t)$ by using different linear combinations of $\mathcal{J}(t)$ and $\mathcal{K}(t)$, depending on the particular values of a and c.

Precisely, condition (1.15) ensures that a suitable linear combination of $\mathcal{I}(t)$, $\mathcal{J}(t)$ and $\mathcal{K}(t)$ is enough to show dispersion for any small solution in the energy space: we will prove that there are parameters $\alpha, \beta \in \mathbb{R}$ and $d_0 > 0$ such that

$$\frac{d}{dt}\left(\mathcal{I}(t) + \alpha\mathcal{J}(t) + \beta\mathcal{K}(t)\right)$$
$$\gtrsim_{a,b,c,d_0} \int e^{-d_0|x|}(u^2 + \eta^2 + (\partial_x u)^2 + (\partial_x \eta)^2)(t, x) dx. \tag{1.19}$$

The choice of parameters in the linear combination $\mathcal{I}(t) + \alpha\mathcal{J}(t) + \beta\mathcal{K}(t)$ is not trivial and we need to change to well-known variables (see [24,25] for its use in the BBM equation [10]), that we refer as "canonical" because of its natural emergence in this type of problems. These new variables are introduced in order to show

positivity of the bilinear form associated to the virial evolution, exactly as in (1.19). Canonical variables are specially useful when dealing with local and nonlocal terms of the form

$$\int e^{-d_0|x|}u^2, \quad \int e^{-d_0|x|}u(1-\partial_x^2)^{-1}u,$$

and similar others, and which in principle have no clear connection or comparison.

The virial technique that we use in this paper is not new. It was inspired in the recent results obtained by the last three authors in [42], which are inspired in foundational works by Martel and Merle [37,38], and Merle and Raphaël [40] for gKdV and NLS equations (see also [23] for another use of virial identities to show decay in the H^1 vicinity of the Zakharov–Kuznetsov solitary wave). An important improvement was the extension of virial identities of this type to the case of wave-like equations, see the works by Kowalczyk, Martel and the second author [27,28]. Then, this approach was successfully extended to the case of the good Boussinesq system in [42]. In this paper, we have followed some of the ideas in [42], especially the introduction of the functional $\mathcal{I}(t)$. However, the main difference between [42] and this paper is that here we need to introduce two additional modifications of the main virial term $\mathcal{I}$ in order to show positivity (in most of the cases), and no Kato-smoothing effect to control higher derivatives is needed. We are also able to prove Theorem 1.4 without any assumption on the parity of the data, leading to a general result in the energy space.

Solitary waves

In addition to long range nonlinearities and modified scattering, the existence of solitary waves is another enemy to decay (in the energy space essentially), and must be taken into account by the conclusions of Theorem 1.4. In [9], the authors investigate the existence of solitary waves for (1.5) coming from ground states, namely solutions of the form

$$u(t,x) = U_\omega(x - \omega t), \quad \eta(t,x) = N_\omega(x - \omega t), \quad \omega \in \mathbb{R}, \tag{1.20}$$

and where (U_ω, N_ω) obey a variational characterization. Among other results, they prove the existence of (nonzero) ground states $(U_\omega, N_\omega) \in H^1 \times H^1$ as long as

$$a, c < 0, \quad b = d, \quad |\omega| < \omega_0, \quad \omega_0 := \begin{cases} \min\{1, \sqrt{ac}\}, & b \neq 0, \\ 1, & b = 0. \end{cases}$$

(Note that the speed ω never reaches the speed of light = 1.) The construction of (U_ω, N_ω) in [9] is involved, and based in stationary, elliptic techniques, and minimization techniques in well-chosen admissible manifolds. See also [19,20,44] for further results on the existence of solitary waves for (1.5).

Although not explicit in general, (U_ω, N_ω) are sometimes explicit. For instance ([26]), for $\omega = 0$ and $a = c < 0$,

$$(U_0, N_0)(x) := \left(\sqrt{2}\, Q\left(\frac{x}{\sqrt{|a|}}\right), -Q\left(\frac{x}{\sqrt{|a|}}\right)\right),$$

$$Q(x) := \frac{3}{2\cosh^2\left(\frac{x}{2}\right)}, \quad Q > 0 \text{ solution of } Q'' - Q + Q^2 = 0,$$

is a standing wave solution of (1.5) (see [26] for other explicit solitary wave solutions, as well as the study of their linear and nonlinear stability). This simple fact also reveals that zero speed solutions are somehow "too

large" to be considered by the hypotheses of Theorem 1.4, and exactly this is the case. Indeed, Theorem 1.4 gives a *dynamical* proof of nonexistence of any small solitary waves with zero speed.

Corollary 1.1 *(Nonexistence of non scattering waves). Under the assumptions of Theorem 1.4, the following hold:*

1. *The abcd Boussinesq system* (1.5) *has no small, zero-speed solitary waves in the form* (1.20).
2. *Any small solitary wave of nonzero speed ω exits the interval $I(t)$ in* (1.9) *in finite time, no matter how small the speed ω is.*
3. *No standing small breather solutions exists for* (1.5).

By standing breather solution we mean a solution $(u, \eta) \in H^1 \times H^1$ of (1.5) which is global and periodic in time. Several dispersive models do have stable and unstable breathers, and their existence are important counterexamples to asymptotic stability of the vacuum solution. Corollary 1.1 discards the existence of such solutions near any bounded neighborhood of zero. See e.g. [2,3,5,6,41,1] and references therein for further details on breather solutions.

Finally, note that Corollary 1.1 shows that Theorem 1.4 is almost sharp (except by a $\log^2 t$ loss), in the sense that solitary waves (1.20) move along lines $x = \omega t + const.$, and they do not decay in the energy space.

Remark 1.9. If the non Hamiltonian case is considered, the global existence of small solutions is not guaranteed. In this situation, we believe that some of the techniques involved in this paper may be useful to improve the current literature. Additionally, the two dimensional case will be part of a forthcoming publication.

Organization of this paper

This paper is organized as follows: Section 2 contains some preliminary results needed along this paper, as well as the proof of Lemma 1.1, first item. Section 3 deals with the Virial identities needed for the proof of Theorem 1.4. Section 4 contains the study of the positivity of the bilinear form appearing from the virial dynamics, and the proof of Lemma 1.1, second item. The purpose of Section 5 is to obtain integrability in time of the local $H^1 \times H^1$ norm of the solution, the main consequence of the virial identity. Section 6 contains energy estimates needed to improve virial estimates from previous sections. Finally, in Section 7 we prove Theorem 1.4.

2. Preliminaries

This section describes some auxiliary results that we will need in the following sections.

2.1. Numerology in the abcd Boussinesq system

Consider the parameters (a, b, c, d) satisfying (1.2) and (1.3) only (note that we are not assuming (1.6)). The purpose of this paragraph is to better understand these conditions, in order to justify some computations in Section 4.

First of all, note that (1.3) and (1.2) imply that

$$b = \frac{1}{6} - \frac{1}{2}(a + c), \tag{2.1}$$

and then

$$b > \frac{1}{6}.$$

This will be the starting point of the following reasoning. Using (1.2) we also obtain

$$b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right) - a, \quad \theta \in [0, 1], \tag{2.2}$$

and

$$b = \frac{1}{2}\left(1 - \theta^2\right) - c, \quad \theta \in [0, 1]. \tag{2.3}$$

Fix $b \in \mathbb{R}$. Let us define sets of pairs (a, c) satisfying the following conditions:

$$
\begin{aligned}
\mathcal{B}_0 &:= \left\{(a, c) \in \mathbb{R}^2 : a < 0, \ c < 0\right\}, \\
\mathcal{B}_1(b) &:= \left\{(a, c) \in \mathbb{R}^2 : a + c = \frac{1}{3} - 2b\right\}, \\
\mathcal{B}_2(b) &:= \left\{(a, c) \in \mathbb{R}^2 : -b - \frac{1}{6} \le a \le -b + \frac{1}{3}\right\}, \\
\mathcal{B}_3(b) &:= \left\{(a, c) \in \mathbb{R}^2 : -b \le c \le -b + \frac{1}{2}\right\}.
\end{aligned}
\tag{2.4}
$$

Note that $\mathcal{B}_1(b)$, $\mathcal{B}_2(b)$ and $\mathcal{B}_3(b)$ reflect the conditions (2.1), (2.2) and (2.3), respectively (see Fig. 1 for details). One can see that the set $\mathcal{B}_1(b)$ describes the line on the (a, c)-plane passing through the points $\left(-b - \frac{1}{6}, -b + \frac{1}{2}\right)$ and $\left(-b + \frac{1}{3}, -b\right)$, which means $\mathcal{B}_1(b) \cap \mathcal{B}_2(b) \cap \mathcal{B}_3(b) \ne \emptyset$ for all $b \in \mathbb{R}$. Moreover,

$$\mathcal{B}_0 \cap \mathcal{B}_1(b) \cap \mathcal{B}_2(b) \cap \mathcal{B}_3(b) \ne \emptyset$$

when $b > \frac{1}{6}$. We resume these findings in the following result.

Lemma 2.1. *There exist (a, b, c, d) such that (1.2) and (1.3) are satisfied if and only if $b > \frac{1}{6}$.*

Note that this result proves Lemma 1.1, item (i).

Remark 2.1. An alternative form for expressing the set $\mathcal{P}_0(b) := \mathcal{B}_0 \cap \mathcal{B}_1(b) \cap \mathcal{B}_2(b) \cap \mathcal{B}_3(b)$ above described is the following 2-parameter description: the set of points (a, b, c) such that

$$(a, b, c) = \left(-\frac{\nu}{2} + \frac{1}{3} - b, \ b, \ \frac{\nu}{2} - b\right), \quad \nu \in [0, 1] \cap \left(\frac{2}{3} - 2b, 2b\right). \tag{2.5}$$
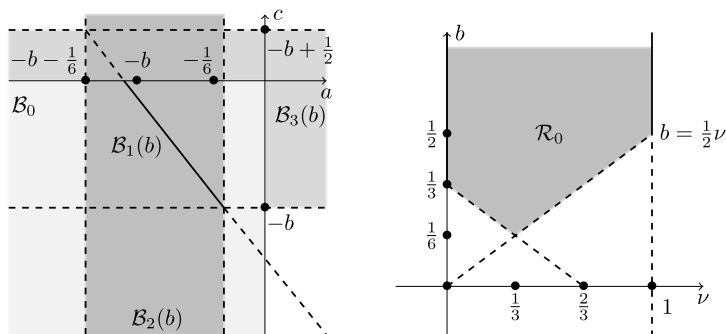
Fig. 1. (*Left.*) The set $\mathcal{B}_0 \cap \mathcal{B}_1(b) \cap \mathcal{B}_2(b) \cap \mathcal{B}_3(b)$ depending on the parameter b, in the case where $b > \frac{1}{6}$. The **continuous** segment of the line $\mathcal{B}_1(b)$ included in $\mathcal{B}_0 \cap \mathcal{B}_2(b) \cap \mathcal{B}_3(b)$ corresponds to the admissible set of parameters (a, c) (depending on b) for which the *abcd* Boussinesq system makes physical sense. (*Right.*) A new representation (in bold) of the set $\mathcal{R}_0$ defined in (2.6) in terms of free parameters (ν, b), see (2.5). Note that each point has associated values (a, c) via formula (2.5), and the set of admissible values makes sense only if $b > \frac{1}{6}$. Note that at $(\nu, b) = (\frac{1}{3}, \frac{1}{6})$, one has $(a, c) = (0, 0)$.

Finally, we introduce the sets

$$\mathcal{P}_0 := \bigcup_{b > 1/6} \mathcal{P}_0(b), \quad \mathcal{R}_0 := \left\{ (\nu, b) \ : \ \nu \in [0, 1] \cap \left(\frac{2}{3} - 2b, 2b \right), \ b > \frac{1}{6} \right\}. \tag{2.6}$$

See Fig. 1, right panel, for more details.

2.2. Canonical variables

It turns out that virial identities for (1.5) are not well-suited for the variables (u, η), but instead better suited for natural canonical variables that appear in models involving the nonlocal operator $(1 - \partial_x^2)^{-1}$.

Definition 2.1 *(Canonical variable).* Let $u = u(x) \in L^2$ be a fixed function. We say that f is canonical variable for u if f uniquely solves the equation

$$f - f_{xx} = u, \quad f \in H^2(\mathbb{R}). \tag{2.7}$$

In this case, we simply denote $f = (1 - \partial_x^2)^{-1} u$.

Canonical variables are standard in equations where the operator $(1 - \partial_x^2)^{-1}$ appears; one of the well-known example is given by the Benjamin–Bona–Mahoney BBM equation, see e.g. [24,25].

Lemma 2.2 *(Equivalence of local L^2 norms). Let f be a canonical variable for $u \in L^2$, as introduced in Definition 2.1. Let ϕ be a smooth, bounded positive weight satisfying $|\phi''| \leq \lambda\phi$ for some small but fixed $0 < \lambda \ll 1$. Then, for any $a_1, a_2, a_3 > 0$, there exist $c_1, C_1 > 0$, depending on a_j $(j = 1, 2, 3)$ and $\lambda > 0$, such that*

$$c_1 \int \phi \, u^2 \leq \int \phi \left(a_1 f^2 + a_2 f_x^2 + a_3 f_{xx}^2 \right) \leq C_1 \int \phi \, u^2. \tag{2.8}$$

Proof. A direct calculation using (2.7) gives the identity

$$\int \phi u^2 = \int \phi \left(f^2 + 2f_x^2 + f_{xx}^2 \right) - \int \phi'' f^2. \tag{2.9}$$

From the property of ϕ, the right-hand side of (2.9) is bounded by

$$\int \phi \left((1 + \lambda) f^2 + 2 f_x^2 + f_{xx}^2 \right),$$

for some $0 < \lambda \ll 1$. Let $\tilde{a} := \min(a_1, a_2, a_3)$ and $c_1 := \frac{\tilde{a}}{2}$. Then, we have

$$\int \phi u^2 \leq \int \phi \left((1 + \lambda) f^2 + 2 f_x^2 + f_{xx}^2 \right)$$
$$\leq c_1^{-1} \int \phi \left(a_1 f^2 + a_2 f_x^2 + a_3 f_{xx}^2 \right).$$

For the reverse part, we have from (2.9) that

$$\int \phi u^2 \geq \int \phi \left((1 - \lambda) f^2 + 2 f_x^2 + f_{xx}^2 \right),$$

for some $0 < \lambda \ll 1$. Let $\tilde{A} := \max(a_1, a_2, a_3)$ and $C_1 := \frac{\tilde{A}}{1 - \lambda}$. Then, a direct calculation yields

$$\int \phi u^2 \geq \int \phi \left((1 - \lambda) f^2 + 2 f_x^2 + f_{xx}^2 \right)$$
$$\geq C_1^{-1} \int \phi \left(a_1 f^2 + a_2 f_x^2 + a_3 f_{xx}^2 \right).$$

The proof is complete. $\square$

The same argument as in the proof of Lemma 2.2 yields the following result.

Lemma 2.3 *(Equivalence of local H^1 norms). Let f be a canonical variable for $u \in H^1$ defined as in Definition 2.1. Let ϕ be a smooth, bounded positive weight satisfying $|\phi''| \leq \lambda \phi$ for some small but fixed $0 < \lambda \ll 1$. Then, for any $d_1, d_2, d_3 > 0$, there exist $c_1, C_1 > 0$ depending on d_j, $j = 1, 2, 3$, and $\lambda > 0$ such that*

$$c_2 \int \phi u_x^2 \leq \int \phi \left(d_1 f_x^2 + d_2 f_{xx}^2 + d_3 f_{xxx}^2 \right) \leq C_2 \int \phi u_x^2. \tag{2.10}$$

2.3. Comparison principle in terms of canonical variables

The following results are well-known in the literature, see El Dika [24] for further details and proofs.

Lemma 2.4 *([24]). The operator $(1 - \partial_x^2)^{-1}$ satisfies the following comparison principle: for any $u, v \in H^1$,*

$$v \leq w \quad \Longrightarrow \quad (1 - \partial_x^2)^{-1} v \leq (1 - \partial_x^2)^{-1} w. \tag{2.11}$$

(2.11) is a consequence of the fact that $(1 - \partial_x^2)^{-1} u \sim e^{-|x|} \star u$. Next result is a consequence of Lemma 2.4 and the Sobolev embedding $u \in H^1 \implies u \in L^\infty$, see the proof of Lemma 2.1 in [24].

Lemma 2.5 *([24]). Suppose that $\phi = \phi(x)$ is such that*

$$(1 - \partial_x^2)^{-1} \phi(x) \lesssim \phi(x), \quad x \in \mathbb{R}, \tag{2.12}$$

for $\phi(x) > 0$ satisfying $|\phi^{(n)}(x)| \lesssim \phi(x)$, $n \geq 0$. Then, for $v, w, h \in H^1$, we have

$$\int \phi^{(n)} v (1 - \partial_x^2)^{-1} (wh)_x \lesssim \|v\|_{H^1} \int \phi(w^2 + w_x^2 + h^2 + h_x^2), \qquad (2.13)$$

and

$$\int \phi^{(n)} v (1 - \partial_x^2)^{-1} (wh) \lesssim \|v\|_{H^1} \int \phi(w^2 + h^2). \qquad (2.14)$$

We will also need modified versions of Lemma 2.5, which will be useful when estimating nonlinear terms in the energy estimate (Section 6).

Lemma 2.6. *Under the same condition as in Lemma 2.5, we have*

$$\int (\phi v_x)_x (1 - \partial_x^2)^{-1} (wh) \lesssim \|v\|_{H^1} \int \phi(w^2 + w_x^2 + h^2 + h_x^2). \qquad (2.15)$$

Proof. The product rule gives

$$\phi v_x = (\phi v)_x - \phi' v \quad \Longrightarrow \quad (\phi v_x)_x = (\phi v)_{xx} - (\phi' v)_x. \qquad (2.16)$$

(2.16) and the integration by parts lead to

$$\begin{aligned}
\text{LHS of (2.15)} &= \int \left((\phi v)_{xx} - (\phi' v)_x \right) (1 - \partial_x^2)^{-1} (wh) \\
&= -\int \phi v w h + \int \phi v (1 - \partial_x^2)^{-1} (wh) \\
&\quad + \int \phi'(x) v (1 - \partial_x^2)^{-1} (w_x h + w h_x).
\end{aligned}$$

Thus, the Cauchy–Schwarz inequality and Lemma 2.5 proves (2.15). $\square$

Lemma 2.7. *Under the same condition as in Lemma 2.5, we have*

$$\int \phi v_x (1 - \partial_x^2)^{-1} (wh)_x \lesssim \|v\|_{H^1} \int \phi(w^2 + w_x^2 + h^2 + h_x^2). \qquad (2.17)$$

Proof. The left-hand side of (2.16) and the integration by parts allow us to get

$$\begin{aligned}
\text{LHS of (2.17)} &= -\int \phi' v (1 - \partial_x^2)^{-1} (w_x h + w h_x) \\
&\quad - \int \phi v (1 - \partial_x^2)^{-1} (wh)_{xx} \\
&= -\int \phi' v (1 - \partial_x^2)^{-1} (w_x h + w h_x) \\
&\quad + \int \phi v w h - \int \phi v (1 - \partial_x^2)^{-1} (wh).
\end{aligned}$$

Thus, the Cauchy–Schwarz inequality and Lemma 2.5 proves (2.17). $\square$

3. Virial functionals

This Section is devoted to the introduction and study of three virial functionals, and their behavior under the $H^1 \times H^1$ flow. Let $\varphi = \varphi(x)$ be a smooth, bounded weight function, to be chosen later. For each $t \in \mathbb{R}$, we consider the following functionals for some φ (to be chosen later):

$$\mathcal{I}(t) := \int \varphi(x)(u\eta + \partial_x u \partial_x \eta)(t, x)dx, \tag{3.1}$$

$$\mathcal{J}(t) := \int \varphi'(x)(\eta \partial_x u)(t, x)dx, \tag{3.2}$$

and

$$\mathcal{K}(t) := \int \varphi'(x)(\partial_x \eta u)(t, x)dx. \tag{3.3}$$

Clearly each functional above is well-defined for $H^1 \times H^1$ functions, as long as the pair $(u, \eta)(t = 0)$ is small in the energy space.

3.1. Virial functional $\mathcal{I}(t)$

Using (1.5) and integration by parts, we have the following result.

Lemma 3.1. *For any $t \in \mathbb{R}$,*

$$
\begin{aligned}
\frac{d}{dt}\mathcal{I}(t) = \ & -\frac{a}{2}\int \varphi' u_x^2 - \frac{c}{2}\int \varphi' \eta_x^2 \\
& -\left(a + \frac{1}{2}\right)\int \varphi' u^2 - \left(c + \frac{1}{2}\right)\int \varphi' \eta^2 \\
& + (1 + a)\int \varphi' u(1 - \partial_x^2)^{-1}u + (1 + c)\int \varphi' \eta(1 - \partial_x^2)^{-1}\eta \\
& -\frac{1}{2}\int \varphi' u^2 \eta + \int \varphi' u(1 - \partial_x^2)^{-1}(u\eta) + \frac{1}{2}\int \varphi' \eta(1 - \partial_x^2)^{-1}(u^2).
\end{aligned}
\tag{3.4}
$$

Proof. We compute:

$$
\begin{aligned}
\frac{d}{dt}\mathcal{I}(t) &= \int \varphi(\eta_t u + \eta u_t + u_{tx}\eta_x + u_x \eta_{tx}) \\
&= \int \varphi(\eta_t - \eta_{txx})u + \int \varphi(u_t - u_{txx})\eta - \int \varphi' \eta u_{tx} - \int \varphi' u \eta_{tx}.
\end{aligned}
$$

Replacing (1.5), and integrating by parts, we get

$$
\begin{aligned}
\frac{d}{dt}\mathcal{I}(t) = \ & \int (\varphi u)_x (au_{xx} + u + u\eta) + \int (\varphi \eta)_x \left(c\eta_{xx} + \eta + \frac{1}{2}u^2\right) \\
& + \int (\varphi' \eta)_x u_t + \int (\varphi' u)_x \eta_t \\
=: \ & I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

First of all, using (1.5),

$$
\begin{aligned}
I_3 &= \int (\varphi' \eta)_x u_t = \int (\varphi' \eta)_{xx} (1 - \partial_x^2)^{-1} \left(c\eta_{xx} + \eta + \frac{1}{2} u^2 \right) \\
&= \int \left((\varphi' \eta)_{xx} - \varphi' \eta \right) (1 - \partial_x^2)^{-1} \left(c\eta_{xx} + \eta + \frac{1}{2} u^2 \right) \\
&\quad + \int \varphi' \eta (1 - \partial_x^2)^{-1} \left(c\eta_{xx} + \eta + \frac{1}{2} u^2 \right) \\
&= - \int \varphi' \eta \left(c\eta_{xx} + \eta + \frac{1}{2} u^2 \right) \\
&\quad + \int \varphi' \eta (1 - \partial_x^2)^{-1} \left(c\eta_{xx} + \eta + \frac{1}{2} u^2 \right).
\end{aligned}
$$

Therefore,

$$
I_2 + I_3 = \int \varphi \eta_x \left(c\eta_{xx} + \eta + \frac{1}{2} u^2 \right) + \int \varphi' \eta (1 - \partial_x^2)^{-1} \left(c\eta_{xx} + \eta + \frac{1}{2} u^2 \right).
$$

Similarly,

$$
\begin{aligned}
I_4 &= - \int \varphi' u \left(a u_{xx} + u + u\eta \right) \\
&\quad + \int \varphi' u (1 - \partial_x^2)^{-1} \left(a u_{xx} + u + u\eta \right),
\end{aligned}
$$

and

$$
I_1 + I_4 = \int \varphi u_x \left(a u_{xx} + u + u\eta \right) + \int \varphi' u (1 - \partial_x^2)^{-1} \left(a u_{xx} + u + u\eta \right).
$$

We conclude that

$$
\begin{aligned}
\frac{d}{dt} \mathcal{I}(t) &= \int \varphi u_x \left(a u_{xx} + u + u\eta \right) + \int \varphi' u (1 - \partial_x^2)^{-1} \left(a u_{xx} + u + u\eta \right) \\
&\quad + \int \varphi \eta_x \left(c\eta_{xx} + \eta + \frac{1}{2} u^2 \right) + \int \varphi' \eta (1 - \partial_x^2)^{-1} \left(c\eta_{xx} + \eta + \frac{1}{2} u^2 \right) \\
&=: \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 + \tilde{I}_4.
\end{aligned}
$$

Now we compute $\tilde{I}_j$. First,

$$
\tilde{I}_1 = -\frac{a}{2} \int \varphi' u_x^2 - \frac{1}{2} \int \varphi' u^2 - \frac{1}{2} \int \varphi' u^2 \eta - \frac{1}{2} \int \varphi u^2 \eta_x.
$$

Second,

$$
\tilde{I}_3 = -\frac{c}{2} \int \varphi' \eta_x^2 - \frac{1}{2} \int \varphi' \eta^2 + \frac{1}{2} \int \varphi \eta_x u^2.
$$

Consequently,

$$
\tilde{I}_1 + \tilde{I}_3 = -\frac{a}{2} \int \varphi' u_x^2 - \frac{c}{2} \int \varphi' \eta_x^2 - \frac{1}{2} \int \varphi' (u^2 + \eta^2) - \frac{1}{2} \int \varphi' u^2 \eta.
$$

On the other hand,

$$
\begin{aligned}
\tilde{I}_2 &= \int \varphi' u (1 - \partial_x^2)^{-1} \left(a u_{xx} + u + u\eta \right) \\
&= a \int \varphi' u (1 - \partial_x^2)^{-1} (u_{xx} - u) \\
&\quad + (1 + a) \int \varphi' u (1 - \partial_x^2)^{-1} u + \int \varphi' u (1 - \partial_x^2)^{-1} (u\eta) \\
&= -a \int \varphi' u^2 + (1 + a) \int \varphi' u (1 - \partial_x^2)^{-1} u + \int \varphi' u (1 - \partial_x^2)^{-1} (u\eta).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\tilde{I}_4 &= \int \varphi' \eta (1 - \partial_x^2)^{-1} \left(c \eta_{xx} + \eta + \frac{1}{2} u^2 \right) \\
&= c \int \varphi' \eta (1 - \partial_x^2)^{-1} (\eta_{xx} - \eta) \\
&\quad + (1 + c) \int \varphi' \eta (1 - \partial_x^2)^{-1} \eta + \frac{1}{2} \int \varphi' \eta (1 - \partial_x^2)^{-1} (u^2) \\
&= -c \int \varphi' \eta^2 + (1 + c) \int \varphi' \eta (1 - \partial_x^2)^{-1} \eta + \frac{1}{2} \int \varphi' \eta (1 - \partial_x^2)^{-1} (u^2).
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
\frac{d}{dt} \mathcal{I}(t) &= \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 + \tilde{I}_4 \\
&= -\frac{a}{2} \int \varphi' u_x^2 - \frac{c}{2} \int \varphi' \eta_x^2 - \frac{1}{2} \int \varphi' (u^2 + \eta^2) - \frac{1}{2} \int \varphi' u^2 \eta \\
&\quad - a \int \varphi' u^2 + (1 + a) \int \varphi' u (1 - \partial_x^2)^{-1} u + \int \varphi' u (1 - \partial_x^2)^{-1} (u\eta) \\
&\quad - c \int \varphi' \eta^2 + (1 + c) \int \varphi' \eta (1 - \partial_x^2)^{-1} \eta + \frac{1}{2} \int \varphi' \eta (1 - \partial_x^2)^{-1} (u^2) \\
&= -\frac{a}{2} \int \varphi' u_x^2 - \frac{c}{2} \int \varphi' \eta_x^2 - \left(a + \frac{1}{2} \right) \int \varphi' u^2 - \left(c + \frac{1}{2} \right) \int \varphi' \eta^2 \\
&\quad + (1 + a) \int \varphi' u (1 - \partial_x^2)^{-1} u + (1 + c) \int \varphi' \eta (1 - \partial_x^2)^{-1} \eta \\
&\quad - \frac{1}{2} \int \varphi' u^2 \eta + \int \varphi' u (1 - \partial_x^2)^{-1} (u\eta) + \frac{1}{2} \int \varphi' \eta (1 - \partial_x^2)^{-1} (u^2).
\end{aligned}
$$

This last equality proves (3.4). $\square$

3.2. Virial functional $\mathcal{J}(t)$

In what follows, we recall the system (1.5)–(1.6) written in the equivalent form

$$
\begin{cases}
\partial_t \eta = a \partial_x u - (1 + a)(1 - \partial_x^2)^{-1} \partial_x u - (1 - \partial_x^2)^{-1} \partial_x (u\eta) \\
\partial_t u = c \partial_x \eta - (1 + c)(1 - \partial_x^2)^{-1} \partial_x \eta - (1 - \partial_x^2)^{-1} \partial_x (\frac{1}{2} u^2).
\end{cases}
\tag{3.5}
$$

Lemma 3.2. *For any* $t \in \mathbb{R}$,

$$
\begin{aligned}
\frac{d}{dt} \mathcal{J}(t) = &\, (1+c) \int \varphi' \eta^2 - c \int \varphi' \eta_x^2 - (1+a) \int \varphi' u^2 + a \int \varphi' u_x^2 \\
&- (1+c) \int \varphi' \eta (1 - \partial_x^2)^{-1} \eta + (1+a) \int \varphi' u (1 - \partial_x^2)^{-1} u \\
&+ (1+a) \int \varphi'' u (1 - \partial_x^2)^{-1} u_x + \frac{c}{2} \int \varphi''' \eta^2 \\
&- \frac{1}{2} \int \varphi' u^2 \eta - \frac{1}{2} \int \varphi' \eta (1 - \partial_x^2)^{-1} \left(u^2 \right) \\
&+ \int \varphi' u (1 - \partial_x^2)^{-1} \left(u\eta \right) + \int \varphi'' u (1 - \partial_x^2)^{-1} (u\eta)_x.
\end{aligned}
\tag{3.6}
$$

Proof. We compute, using (3.5),

$$
\begin{aligned}
\frac{d}{dt} \mathcal{J}(t) &= \int \varphi' \left(\eta_t u_x + \eta u_{xt} \right) \\
&= \int \varphi' u_x \left(a u_x - (1+a)(1 - \partial_x^2)^{-1} u_x - (1 - \partial_x^2)^{-1}(u\eta)_x \right) \\
&\quad + \int \varphi' \eta \left(c \eta_{xx} - (1+c)(1 - \partial_x^2)^{-1} \eta_{xx} - \frac{1}{2}(1 - \partial_x^2)^{-1}(u^2)_{xx} \right) \\
&=: J_1 + J_2.
\end{aligned}
$$

We first deal with J_1. The integration by parts yields

$$
\begin{aligned}
J_1 &= a \int \varphi' u_x^2 - (1+a) \int \varphi' u_x (1 - \partial_x^2)^{-1} u_x - \int \varphi' u_x (1 - \partial_x^2)^{-1}(u\eta)_x \\
&= a \int \varphi' u_x^2 + (1+a) \int \varphi' u (1 - \partial_x^2)^{-1} u_{xx} \\
&\quad + (1+a) \int \varphi'' u (1 - \partial_x^2)^{-1} u_x \\
&\quad + \int \varphi' u (1 - \partial_x^2)^{-1}(u\eta)_{xx} + \int \varphi'' u (1 - \partial_x^2)^{-1}(u\eta)_x.
\end{aligned}
\tag{3.7}
$$

We use

$$
\int \psi w (1 - \partial_x^2)^{-1} z_{xx} = - \int \psi w z + \int \psi w (1 - \partial_x^2)^{-1} z
\tag{3.8}
$$

for the second and fourth terms, in the right-hand side of (3.7), to obtain

$$
\begin{aligned}
J_1 = &\, -(1+a) \int \varphi' u^2 + a \int \varphi' u_x^2 \\
&+ (1+a) \int \varphi' u (1 - \partial_x^2)^{-1} u + (1+a) \int \varphi'' u (1 - \partial_x^2)^{-1} u_x \\
&- \int \varphi' u^2 \eta + \int \varphi' u (1 - \partial_x^2)^{-1}(u\eta) + \int \varphi'' u (1 - \partial_x^2)^{-1}(u\eta)_x.
\end{aligned}
\tag{3.9}
$$

For J_2, using the integration by parts and (3.8) with the identity $f f_{xx} = \frac{1}{2}(f^2)_{xx} - f_x^2$, yields

$$
\begin{aligned}
J_2 &= c \int \varphi' \eta \eta_{xx} - (1+c) \int \varphi' \eta (1 - \partial_x^2)^{-1} \eta_{xx} \\
&\quad - \frac{1}{2} \int \varphi' \eta (1 - \partial_x^2)^{-1} (u^2)_{xx} \\
&= (1+c) \int \varphi' \eta^2 - c \int \varphi' \eta_x^2 + \frac{c}{2} \int \varphi''' \eta^2 \\
&\quad - (1+c) \int \varphi' \eta (1 - \partial_x^2)^{-1} \eta \\
&\quad + \frac{1}{2} \int \varphi' u^2 \eta - \frac{1}{2} \int \varphi' \eta (1 - \partial_x^2)^{-1}(u^2).
\end{aligned}
\tag{3.10}
$$

By collecting all (3.9) and (3.10), we have (3.6). $\quad\square$

3.3. Virial functional $\mathcal{K}(t)$

In view of the *abcd* system (3.5), one can realize that both equations have a sort of weakly symmetric structure (up to constants a, c, and nonlinearities). A slight modification in Lemma 3.2 (using the replacements $u \leftrightarrow \eta$, $a \leftrightarrow c$ and $u\eta \leftrightarrow \frac{1}{2}u^2$) provides the following result for the virial functional $\mathcal{K}(t)$. The interested reader may check the details.

Lemma 3.3. *For any $t \in \mathbb{R}$,*

$$
\begin{aligned}
\frac{d}{dt}\mathcal{K}(t) = &-(1+c) \int \varphi' \eta^2 + c \int \varphi' \eta_x^2 + (1+a) \int \varphi' u^2 - a \int \varphi' u_x^2 \\
&+ (1+c) \int \varphi' \eta (1 - \partial_x^2)^{-1} \eta - (1+a) \int \varphi' u (1 - \partial_x^2)^{-1} u \\
&+ (1+c) \int \varphi'' \eta (1 - \partial_x^2)^{-1} \eta_x + \frac{a}{2} \int \varphi''' u^2 \\
&- \int \varphi' u (1 - \partial_x^2)^{-1} (u\eta) + \frac{1}{2} \int \varphi' \eta (1 - \partial_x^2)^{-1} (u^2) \\
&+ \frac{1}{2} \int \varphi' u^2 \eta + \frac{1}{2} \int \varphi'' \eta (1 - \partial_x^2)^{-1} (u^2)_x.
\end{aligned}
\tag{3.11}
$$

3.4. Modified virial

Now we construct a global virial from a linear combination of $\mathcal{I}(t)$, $\mathcal{J}(t)$ and $\mathcal{K}(t)$.

For α and β real numbers, we define the modified virial

$$
\mathcal{H}(t) := \mathcal{H}_{\alpha,\beta}(t) := \mathcal{I}(t) + \alpha \mathcal{J}(t) + \beta \mathcal{K}(t).
\tag{3.12}
$$

Thanks to Lemmas 3.1, 3.2 and 3.3, we obtain the following direct consequence:

Proposition 3.1 *(Decomposition of $\frac{d}{dt}\mathcal{H}(t)$).* *Let η and u satisfy (3.5). For any $\alpha, \beta \in \mathbb{R}$ and any $t \in \mathbb{R}$, we have the decomposition*

$$
\frac{d}{dt}\mathcal{H}(t) = \mathcal{Q}(t) + \mathcal{SQ}(t) + \mathcal{NQ}(t),
\tag{3.13}
$$

where $\mathcal{Q}(t) = \mathcal{Q}[u, \eta](t)$ is the quadratic form

$$\begin{aligned} \mathcal{Q}(t) := & \left((1+c)(\alpha - \beta - 1) + \frac{1}{2} \right) \int \varphi' \eta^2 + c \left(\beta - \alpha - \frac{1}{2} \right) \int \varphi' \eta_x^2 \\ & + \left((1+a)(\beta - \alpha - 1) + \frac{1}{2} \right) \int \varphi' u^2 + a \left(\alpha - \beta - \frac{1}{2} \right) \int \varphi' u_x^2 \\ & + (1+c)(\beta - \alpha + 1) \int \varphi' \eta (1 - \partial_x^2)^{-1} \eta \\ & + (1+a)(\alpha - \beta + 1) \int \varphi' u (1 - \partial_x^2)^{-1} u, \end{aligned}$$

(3.14)

$\mathcal{SQ}(t)$ represents lower order quadratic terms not included in $\mathcal{Q}(t)$:

$$\begin{aligned} \mathcal{SQ}(t) := & \beta(1+c) \int \varphi'' \eta (1 - \partial_x^2)^{-1} \eta_x + \alpha(1+a) \int \varphi'' u (1 - \partial_x^2)^{-1} u_x \\ & + \frac{\alpha c}{2} \int \varphi''' \eta^2 + \frac{\beta a}{2} \int \varphi''' u^2, \end{aligned}$$

(3.15)

and $\mathcal{NQ}(t)$ are truly cubic order terms or higher:

$$\begin{aligned} \mathcal{NQ}(t) := & \frac{1}{2}(\beta - \alpha - 1) \int \varphi' u^2 \eta + \frac{1}{2}(\beta - \alpha + 1) \int \varphi' \eta (1 - \partial_x^2)^{-1} (u^2) \\ & + (\alpha - \beta + 1) \int \varphi' u (1 - \partial_x^2)^{-1} (u\eta) + \frac{\beta}{2} \int \varphi'' \eta (1 - \partial_x^2)^{-1} (u^2)_x \\ & + \alpha \int \varphi'' u (1 - \partial_x^2)^{-1} (u\eta)_x. \end{aligned}$$

(3.16)

Remark 3.1. Note that all members in $\mathcal{SQ}(t)$ (3.15) contains terms of the form $\varphi^{(n)}(x)$, $n = 2, 3$, and it makes each term be as small as we want by choosing an appropriate weight function and using rescaling arguments. Also, all terms in $\mathcal{NQ}(t)$ will not be of great importance, because they will be bounded by quantities smaller than any term in $\mathcal{Q}(t)$.

4. Estimates in canonical variables and positivity

4.1. Passage to canonical variables

We focus on the quadratic form $\mathcal{Q}(t)$ in (3.14). We introduce canonical variables for u and η (see Definition 2.1) as follows:

$$f := (1 - \partial_x^2)^{-1} u \quad \text{and} \quad g := (1 - \partial_x^2)^{-1} \eta. \tag{4.1}$$

Note that, for $u, \eta \in H^1$, one has f and g in H^3. A direct calculation shows the following key relationships between f and u (resp. g and η), see also Lemmas 2.2 and 2.3 for similar statements.

Lemma 4.1. *One has*

$$\int \varphi' u^2 = \int \varphi' \left(f^2 + 2 f_x^2 + f_{xx}^2 \right) - \int \varphi''' f^2, \tag{4.2}$$

$$\int \varphi' u_x^2 = \int \varphi' \left(f_x^2 + 2 f_{xx}^2 + f_{xxx}^2 \right) - \int \varphi''' f_x^2 \tag{4.3}$$

and

$$\int \varphi' u (1-\partial_x^2)^{-1} u = \int \varphi' \left(f^2 + f_x^2\right) - \frac{1}{2} \int \varphi''' f^2. \tag{4.4}$$

Using (4.2)–(4.4), we can rewrite the quadratic form $\mathcal{Q}(t)$ as follows:

Lemma 4.2. *Let f and g be canonical variables of u and η as in (4.1). Consider the quadratic form $\mathcal{Q}(t)$ given in (3.14). Then we have*

$$\begin{aligned}
\mathcal{Q}(t) &= \int \varphi' \left(A_1 f^2 + A_2 f_x^2 + A_3 f_{xx}^2 + A_4 f_{xxx}^2\right) \\
&\quad + \int \varphi' \left(B_1 g^2 + B_2 g_x^2 + B_3 g_{xx}^2 + B_4 g_{xxx}^2\right) \\
&\quad + \int \varphi''' \left(D_{11} f^2 + D_{12} f_x^2 + D_{21} g^2 + D_{22} g_x^2\right),
\end{aligned} \tag{4.5}$$

where

$$A_1 = B_1 = \frac{1}{2} > 0, \tag{4.6}$$

$$A_2 = \beta - \alpha - \frac{3a}{2}, \qquad B_2 = \alpha - \beta - \frac{3c}{2}, \tag{4.7}$$

$$A_3 = (1-a)(\beta - \alpha) - 2a - \frac{1}{2}, \qquad B_3 = (1-c)(\alpha - \beta) - 2c - \frac{1}{2}, \tag{4.8}$$

$$A_4 = a\left(\alpha - \beta - \frac{1}{2}\right), \qquad B_4 = c\left(\beta - \alpha - \frac{1}{2}\right), \tag{4.9}$$

and

$$\begin{aligned}
D_{11} &= -\frac{1}{2}(1+a)(\beta - \alpha - 1) - \frac{1}{2}, \qquad D_{12} = -a\left(\alpha - \beta - \frac{1}{2}\right), \\
D_{21} &= -\frac{1}{2}(1+c)(\alpha - \beta - 1) - \frac{1}{2}, \qquad D_{22} = -c\left(\beta - \alpha - \frac{1}{2}\right).
\end{aligned} \tag{4.10}$$

Proof. Direct from the substitution of (4.2)–(4.4) into (3.14). □

Remark 4.1. A slight modification of (4.4) also allows us to rewrite the first two terms in $\mathcal{SQ}(t)$ (3.15) as follows:

$$\beta(1+c) \int \varphi'' \eta (1-\partial_x^2)^{-1} \eta_x = -\frac{1}{2}\beta(1+c) \int \varphi''' (g^2 - g_x^2), \tag{4.11}$$

and

$$\alpha(1+a) \int \varphi'' u (1-\partial_x^2)^{-1} u_x = -\frac{1}{2}\alpha(1+a) \int \varphi''' (f^2 - f_x^2). \tag{4.12}$$

These terms, however, will be negligible compared with the terms in $\mathcal{Q}(t)$.

4.2. Positivity

We want to study positivity (or negativity) properties of the bilinear form $\mathcal{Q}(t)$, namely, to decide under which conditions $\mathcal{Q}(t)$ in (4.5) has a unique definite sign.

It turns out that A_1 and B_1 in (4.6) are positive, so the only possibility is to obtain $\mathcal{Q}(t)$ positive definite. Then, a sufficient condition for getting this goal is to impose

$$A_k > 0, \quad B_k > 0, \quad k = 2, 3, 4. \tag{4.13}$$

Note that the coefficients D_k may have any possible sign, this is because we will assume that φ''' is as smaller as we want, compared to φ'.

The following lemma characterizes condition (4.13) in terms of the *dispersion-like parameters* introduced in Definition 1.1.

Lemma 4.3 (*Positivity vs. dispersion*). *Let $a, c < 0$. Then, there exist parameters $(\alpha, \beta) \in \mathbb{R}^2$, such that $A_k > 0$ and $B_k > 0$ for $k = 2, 3, 4$ if and only if (a, c) are dispersion-like parameters. That is to say,*

$$3(a + c) + 2 - 8ac < 0. \tag{4.14}$$

Later we will prove (see Lemma 4.4) that (4.14) is satisfied provided $b > \frac{2}{9}$. See Fig. 4 for a picture of the region of pairs $(a, c) \in \mathbb{R}^2$ for which (4.14) is satisfied, in terms of the parameter $b > \frac{2}{9}$.

Proof of Lemma 4.3. First of all, we have from (4.9) and the (negative) signs of a, c that A_4 and B_4 are positive whenever (α, β) belongs to the set

$$\mathcal{A}_4 := \left\{ (\alpha, \beta) \in \mathbb{R}^2 : \alpha - \frac{1}{2} < \beta < \alpha + \frac{1}{2} \right\}. \tag{4.15}$$

This is a set consisting of a diagonal band around zero, see Fig. 2. Note that $\mathcal{A}_4$ is never empty, and $(0, 0)$ is always included (recall that $(0, 0)$ corresponds to the case where $\mathcal{J}$ and $\mathcal{K}$ are not included (nor necessary) in $\mathcal{H}$, see (3.12)).

From (4.7), we see that both A_2 and B_2 are positive if (α, β) belongs to the set

$$\mathcal{A}_2 := \left\{ (\alpha, \beta) \in \mathbb{R}^2 : \alpha + \frac{3a}{2} < \beta < \alpha - \frac{3c}{2} \right\}. \tag{4.16}$$

(See Fig. 2.) This set makes sense if and only if

$$\frac{3a}{2} < -\frac{3c}{2} \iff a + c < 0,$$

which is a direct consequence of the fact that $a < 0$ and $c < 0$. Hence, always the set $\mathcal{A}_2$ is nonempty and $(0, 0)$ is contained. Note additionally that $\frac{3a}{2} < 0$ and $-\frac{3c}{2} > 0$, so the boundary of $\mathcal{A}_2$ never crosses the origin. However, depending on the values of a and c, they might contain, be contained, or not contain $\mathcal{A}_4$.

Finally, we deal with the positivity of the terms A_3 and B_3 in (4.8). It is not difficult to show that $A_3 > 0$ and $B_3 > 0$ whenever (α, β) belongs to the set

$$\mathcal{A}_3 := \left\{ (\alpha, \beta) \in \mathbb{R}^2 : \alpha + \frac{1 + 4a}{2(1 - a)} < \beta < \alpha - \frac{1 + 4c}{2(1 - c)} \right\}. \tag{4.17}$$
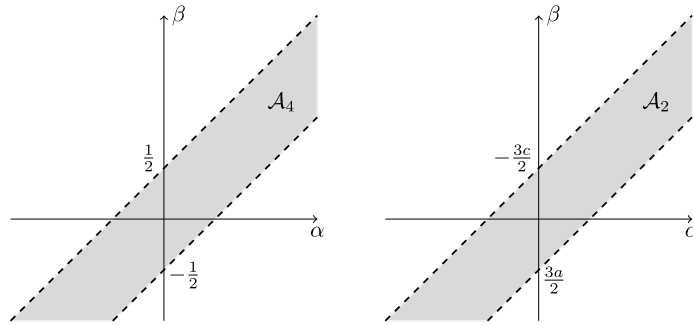
Fig. 2. (*Left.*) The set $\mathcal{A}_4$ defined in (4.15). Note that $(0,0)$ is always included in $\mathcal{A}_4$, meaning that, for having A_4 and B_4 both positive, $\mathcal{J}$ and $\mathcal{K}$ are not needed. (*Right.*) The set $\mathcal{A}_2$ defined in (4.16). This set may or may not be contained in $\mathcal{A}_4$, depending on the values of a and c. It also may be the case that $\mathcal{A}_2$ contains $\mathcal{A}_4$. Its boundary never crosses the origin, and it is not necessarily symmetric with respect to the line $\beta = \alpha$. Therefore, for having $A_2, B_2 > 0$, both $\mathcal{J}$ and $\mathcal{K}$ are not needed.
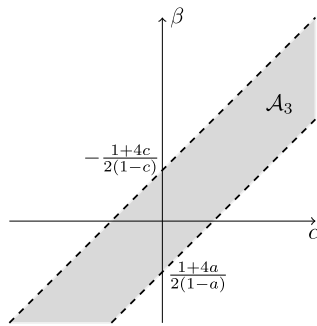

Fig. 3. The set $\mathcal{A}_3$ defined in (4.17). Contrary to other sets $\mathcal{A}_4$ and $\mathcal{A}_2$, this set *may not contain* the point $(0,0)$, depending on the sign of one of the terms $\frac{1+4a}{2(1-a)}$ or $-\frac{1+4c}{2(1-c)}$. In this case, the perturbations $\mathcal{J}$ and $\mathcal{K}$ in (3.12) are fundamental and necessary.

(See Fig. 3.) Note that the set $\mathcal{A}_3$ is well-defined, since $a, c < 0$. We know that $\mathcal{A}_3 \neq \emptyset$ if and only if

$$\frac{1+4a}{2(1-a)} < -\frac{1+4c}{2(1-c)},$$

which is nothing but the dispersion-like condition (4.14). Therefore, we have $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ nonempty.

It remains to show $\mathcal{A}_2 \cap \mathcal{A}_3 \cap \mathcal{A}_4 \neq \emptyset$, namely there exist parameters (α, β) such that (4.13) is satisfied. First of all, the conditions $a, c < 0$ guarantee

$$\frac{3a}{2} < \frac{1}{2} \qquad \text{and} \qquad -\frac{1}{2} < -\frac{3c}{2},$$

which imply $\mathcal{A}_4 \cap \mathcal{A}_2 \neq \emptyset$ without any further conditions on a and c. On the other hand, $\mathcal{A}_4 \cap \mathcal{A}_3 \neq \emptyset$ whenever

$$\frac{1+4a}{2(1-a)} < \frac{1}{2} \qquad \text{and} \qquad -\frac{1}{2} < -\frac{1+4c}{2(1-c)}. \tag{4.18}$$

The condition $a, c < 0$ immediately implies (4.18), and hence $\mathcal{A}_4 \cap \mathcal{A}_3 \neq \emptyset$ without any further conditions on a and c.

Now we deal with the proof of $\mathcal{A}_2 \cap \mathcal{A}_3 \cap \mathcal{A}_4 \neq \emptyset$. Note that each set $\mathcal{A}_i$, $i = 2, 3, 4$, describes an unbounded strip on (α, β)-plane with "the same slope". Hence, it suffices to show that

$$\mathcal{A}_2 \cap \mathcal{A}_3 \neq \emptyset \ \wedge \ \mathcal{A}_2 \cap \mathcal{A}_4 \neq \emptyset \ \wedge \ \mathcal{A}_3 \cap \mathcal{A}_4 \neq \emptyset. \tag{4.19}$$

Indeed, if $\mathcal{A}_2 \cap \mathcal{A}_3 = \mathcal{A}_2$ (or $\mathcal{A}_3$), (4.19) immediately implies $\mathcal{A}_2 \cap \mathcal{A}_3 \cap \mathcal{A}_4 \neq \emptyset$. Hence, suppose that $\mathcal{A}_2 \nsubseteq \mathcal{A}_3$ and $\mathcal{A}_3 \nsubseteq \mathcal{A}_2$. In this case $\mathcal{A}_2 \cap \mathcal{A}_3$ describes a new band constructed by upper and lower edges from $\mathcal{A}_2$ and $\mathcal{A}_3$, respectively, (or vice versa). Without loss of generality, we assume the new band is constructed by upper edge from $\mathcal{A}_2$ and lower edge from $\mathcal{A}_3$. On the other hand, $\mathcal{A}_2 \cap \mathcal{A}_4 \neq \emptyset$ and $\mathcal{A}_3 \cap \mathcal{A}_4 \neq \emptyset$ (and the open character of each involved set) require that the upper edge of $\mathcal{A}_4$ should be positioned above the lower edge of $\mathcal{A}_3$, and the lower edge of $\mathcal{A}_4$ should be positioned below the upper edge of $\mathcal{A}_2$, respectively. This exactly implies $(\mathcal{A}_2 \cap \mathcal{A}_3) \cap \mathcal{A}_4 \neq \emptyset$. This is the only possible case, because all $\mathcal{A}_j$ describe bands with the same slope. Therefore, we only need to check (4.19).

Let us prove the remaining part of (4.19), namely $\mathcal{A}_2 \cap \mathcal{A}_3 \neq \emptyset$. This happens if

$$\frac{1+4a}{2(1-a)} < -\frac{3c}{2} \quad \text{and} \quad \frac{3a}{2} < -\frac{1+4c}{2(1-c)}. \tag{4.20}$$

A direct calculation under $a, c < 0$ yields that (4.20) is equivalent to the conditions

$$3ac > 1 + 4a + 3c, \quad 3ac > 1 + 3a + 4c. \tag{4.21}$$

On the (a,c)-plane, we can see that the region of (a,c) satisfying (4.21) and $a, c < 0$ covers the region of (a,c) satisfying (4.14). Indeed, consider the following two equations

$$8ac - 2 - 3(a+c) = 0 \quad \text{and} \quad 3ac - 1 - 4a - 3c = 0. \tag{4.22}$$

Note that if $a < 0$, we know

$$8a - 3 \neq 0 \ (<0), \quad \text{and} \quad 3a - 3 \neq 0 \ (<0). \tag{4.23}$$

For $a < 0$, (4.22) can be expressed as implicit function formulas of c in terms of a as follows:

$$c = \Gamma_1(a) := \frac{3a+2}{8a-3} \quad \text{and} \quad c = \Gamma_2(a) := \frac{4a+1}{3a-3}.$$

It is known that if $\Gamma_1(a) < \Gamma_2(a)$, the region of $c < \Gamma_2(a)$ covers the region of $c < \Gamma_1(a)$ on (a,c)-plane, and hence the region of (a,c) satisfying (4.14) is contained to the region of (a,c) satisfying $3ac - 1 - 4a - 3c > 0$ for $a < 0$ due to (4.23). A straightforward calculation indeed gives

$$\frac{3a+2}{8a-3} < \frac{4a+1}{3a-3} \iff 23a^2 - a + 3 > 0, \tag{4.24}$$

for all $a < 0$. A similar argument holds for $3ac > 1 + 3a + 4c$.

Thus, if $a < 0$ and $c < 0$ satisfy (4.14), we can always choose $(\alpha, \beta) \in \mathcal{A}_2 \cap \mathcal{A}_3 \cap \mathcal{A}_4$ such that $A_k > 0$ and $B_k > 0$, $k = 1, 2, 3, 4$. The proof is complete. $\square$

In what follows we study whether or not condition (4.14) can be satisfied for different values of $b > \frac{1}{6}$.

Going back to the original parameters a and b from (1.2) and (1.3) (recall that a and c were changed to a/b and c/b respectively), (4.14) can be rewritten as the fact that the set

$$\mathcal{B}_4(b) := \left\{ (a,c) \in \mathbb{R}^2 \ : \ 3b(a+c) + 2b^2 - 8ac < 0 \right\},$$

is nonempty.

Recall the sets $\mathcal{B}_0, \mathcal{B}_1(b), \mathcal{B}_2(b)$ and $\mathcal{B}_3(b)$ defined in (2.4), and Fig. 1 and Lemma 2.1, which characterize the values of b for which the $abcd$ system makes sense. In the following computations, we assume $b > \frac{1}{6}$.
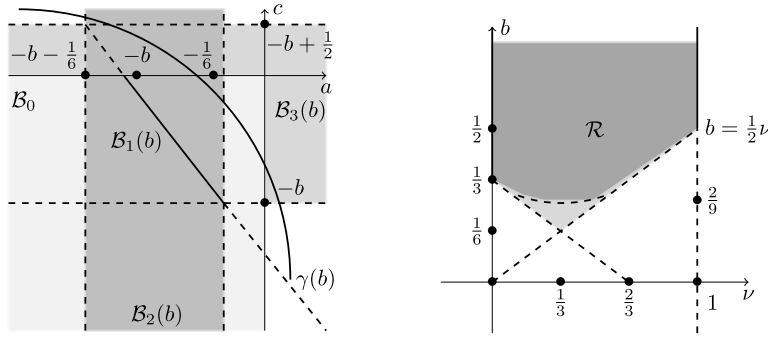
Fig. 4. (*Left.*) Schematic representation of the action of the set $\mathcal{B}_4(b)$ in Fig. 1, in the case where $b > \frac{2}{9} \sim 0.22 > \frac{1}{6} \sim 0.17$. The region *below* the continuous curve $\gamma(b)$ given by the equation $c = -b\left(\frac{2b+3a}{3b-8a}\right)$ represents the admissible (a, c) given in $\mathcal{B}_4(b)$. Note that (a, c) cannot be arbitrarily small. (*Right.*) The dark region $\mathcal{R}$ (subset of $\mathcal{R}_0$ in Fig. 1, right panel) represents all the pairs (ν, b) for which $\mathcal{A}_3$ is nonempty, and therefore Theorem 1.4 is valid. Note that $b = \frac{2}{9}$ represents the bottom of this set, part of an hyperbola, see Appendix B for more details. Probably Theorem 1.4 is still valid for a small portion of the remaining admissible region (light shadowed) below $b = \frac{2}{9}$, but a more involved proof is needed. Finally note that at $(\nu, b) = (\frac{1}{3}, \frac{2}{9})$, one has $(a, c) = (-\frac{1}{18}, -\frac{1}{18})$.

Let $\mathcal{P}_b := \mathcal{B}_0 \cap \mathcal{B}_1(b) \cap \mathcal{B}_2(b) \cap \mathcal{B}_3(b) \cap \mathcal{B}_4(b)$. As we saw in Subsection 2.1, $\mathcal{B}_0 \cap \mathcal{B}_1(b) \cap \mathcal{B}_2(b) \cap \mathcal{B}_3(b)$ represents a diagonal left on (a, c)-plane. Moreover, $\mathcal{B}_4(b)$ describes the region below the continuous curve $\gamma(b)$ given by the equation

$$c = -b\left(\frac{2b+3a}{3b-8a}\right), \quad a \neq \frac{3}{8}b > 0,$$

see Fig. 4 (recall that $a, c < 0$ and that $c = -\frac{2}{3}b$ at $a = 0$ is always above the minimal value $c = -b$ of $\mathcal{B}_1(b)$). Hence, a sufficient condition for having $\mathcal{P}_b \neq \emptyset$ is that the curve $\gamma(b)$ intersects the segment $\mathcal{B}_1(b)$ at two different points, which is equivalent to the following equations having two roots for (a, c)

$$\begin{cases} a + c = \frac{1}{3} - 2b \\ c = -b\left(\frac{2b+3a}{3b-8a}\right). \end{cases} \tag{4.25}$$

Solving (4.25) is reduced to $3456b^2 - 1056b + 64 > 0$, which gives $b < \frac{1}{12} \ (< \frac{1}{6})$ or $b > \frac{2}{9} \ (> \frac{1}{6})$. Thus, we conclude that $\mathcal{P}_b \neq \emptyset$ if $b > \frac{2}{9}$. So far, we have proved

Lemma 4.4 (*Sufficient condition for having dispersion-like parameters*). We have $\mathcal{P}_b = \mathcal{B}_0 \cap \mathcal{B}_1(b) \cap \mathcal{B}_2(b) \cap \mathcal{B}_3(b) \cap \mathcal{B}_4(b) \neq \emptyset$ provided $b > \frac{2}{9}$. Moreover, this set represents a segment of line in the (a, c) variables.

This result proves the second item in Lemma 1.1.

In Appendix B we will prove that $\mathcal{P}_b$ follows a precise parametric characterization. Finally, let us define

$$\mathcal{P} := \mathcal{P}((a, b, c)) = \bigcup_{b > \frac{2}{9}} \mathcal{P}_b, \qquad \mathcal{R} := \{(\nu, b) \in \mathcal{R}_0 \ : \ (a, b, c) \in \mathcal{P}_b \neq \emptyset\}. \tag{4.26}$$

This set $\mathcal{P}$ describes the parameter condition of (a, b, c), which allows both well-defined model in physical sense and dispersive phenomena in mathematical sense.

So far, we have proved the following result:

Lemma 4.5 (*Positivity of the quadratic form $\mathcal{Q}(t)$*). Let $a, c < 0$ satisfying (1.6). Under the dispersion-like condition (4.14) on (a, b, c), there exist parameters $(\alpha, \beta) \in \mathbb{R}^2$ (not necessarily different from (0,0)), such that $\mathcal{H}(t) = \mathcal{I}(t) + \alpha\mathcal{J}(t) + \beta\mathcal{K}(t)$ obeys the following decomposition for its time derivative:

$$\frac{d}{dt}\mathcal{H}(t) = \mathcal{Q}(t) + \mathcal{S}\mathcal{Q}(t) + \mathcal{N}\mathcal{Q}(t), \tag{4.27}$$

where

1. $\mathcal{Q}(t)$ *is given by the expression*

$$\mathcal{Q}(t) = \int \varphi'\left(A_1 f^2 + A_2 f_x^2 + A_3 f_{xx}^2 + A_4 f_{xxx}^2\right)$$
$$+ \int \varphi'\left(B_1 g^2 + B_2 g_x^2 + B_3 g_{xx}^2 + B_4 g_{xxx}^2\right) \tag{4.28}$$
$$+ \int \varphi'''\left(D_{11} f^2 + D_{12} f_x^2 + D_{21} g^2 + D_{22} g_x^2\right),$$

  for $f = (1 - \partial_x^2)^{-1} u$ *and* $g = (1 - \partial_x^2)^{-1}\eta$.
2. **Positivity.** *We have* $A_i, B_j > 0$, *and* $D_{ij} \in \mathbb{R}$.
3. $\mathcal{S}\mathcal{Q}(t), \mathcal{N}\mathcal{Q}(t)$ *are given in* (3.15)–(3.16).

5. Integrability in time

The purpose of this Section is to obtain an integrability-in-time estimate for the terms appearing in Lemma 4.5. This amounts to estimate the various terms in (4.27) using the stability of the zero solution.

Proposition 5.1 *(Decay in compact intervals). Let* (a, b, c) *be* dispersion-like *parameters defined as in Definition 1.1. Let* $(u, \eta)(t)$ *be* $H^1 \times H^1$ *global solutions to* (1.5) *such that* (1.16) *holds. Then, we have for large (but fixed)* $\lambda \gg 1$ *that*

$$\int\limits_2^\infty \int \operatorname{sech}^2\left(\frac{x}{\lambda}\right) \left(u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2\right)(t, x)\, dx\, dt \lesssim \lambda \varepsilon^2. \tag{5.1}$$

As an immediate consequence, there exists an increasing sequence of time $\{t_n\}$ $(t_n \to \infty$ *as* $n \to \infty)$ *such that*

$$\int \operatorname{sech}^2\left(\frac{x}{\lambda}\right) \left(u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2\right)(t_n, x)\, dx \longrightarrow 0 \ \text{as} \ n \to \infty. \tag{5.2}$$

Remark 5.1. Note that estimate (5.1) completes the proof of (1.18).

Proof of Proposition 5.1. Let us choose $\varphi(x) = \lambda \tanh\left(\frac{x}{\lambda}\right)$ in Lemma 4.5, for $\lambda > 1$ to be chosen later. Note that

$$\varphi'(x) = \operatorname{sech}^2\left(\frac{x}{\lambda}\right), \quad |\varphi''(x)| \le \frac{2}{\lambda}\operatorname{sech}^2\left(\frac{x}{\lambda}\right) = \frac{2}{\lambda}\varphi'(x),$$
$$\text{and} \quad |\varphi'''(x)| \le \frac{1}{\lambda^2}\operatorname{sech}^2\left(\frac{x}{\lambda}\right) = \frac{1}{\lambda^2}\varphi'(x). \tag{5.3}$$

From Lemma 4.5, choosing a sufficiently large $\lambda > 1$, the rest terms in (4.28), characterized by the weight φ''', cannot change the sign of A_k and B_k, $k = 1, 2, 3, 4$, due to (5.3). Therefore, they can be easily absorbed in (4.28):

$$\mathcal{Q}(t) \geq \frac{3}{4} \int \varphi' \left(A_1 f^2 + A_2 f_x^2 + A_3 f_{xx}^2 + A_4 f_{xxx}^2 \right)$$
$$+ \frac{3}{4} \int \varphi' \left(B_1 g^2 + B_2 g_x^2 + B_3 g_{xx}^2 + B_4 g_{xxx}^2 \right).$$

Similarly, $\mathcal{SQ}(t)$ in (3.15) can be characterized in terms of canonical variables using e.g. (4.11) and (4.12). Once again, because of the weight terms φ'' and φ''' and (5.3), these terms are negligible with respect to the leading terms in (4.28). We conclude from (4.27),

$$\frac{d}{dt} \mathcal{H}(t) \geq \frac{1}{2} \int \varphi' \left(A_1 f^2 + A_2 f_x^2 + A_3 f_{xx}^2 + A_4 f_{xxx}^2 \right)$$
$$+ \frac{1}{2} \int \varphi' \left(B_1 g^2 + B_2 g_x^2 + B_3 g_{xx}^2 + B_4 g_{xxx}^2 \right) + \mathcal{NQ}(t).$$

Hence, Lemmas 2.2 and 2.3 give

$$\frac{d}{dt} \mathcal{H}(t) \geq C_1 \int \operatorname{sech}^2 \left(\frac{x}{\lambda} \right) u^2 + C_2 \int \operatorname{sech}^2 \left(\frac{x}{\lambda} \right) (\partial_x u)^2$$
$$+ C_3 \int \operatorname{sech}^2 \left(\frac{x}{\lambda} \right) \eta^2 + C_4 \int \operatorname{sech}^2 \left(\frac{x}{\lambda} \right) (\partial_x \eta)^2$$
$$+ \mathcal{NQ}(t),$$

for large $\lambda \gg 1$ and some $C_j > 0$, $j = 1, 2, 3, 4$.

On the other hand, Lemma 2.5 easily gives

$$\mathcal{NQ}(t) \lesssim \left(\|u\|_{H^1} + \|\eta\|_{H^1} \right) \int \varphi'(u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2).$$

This estimate, together with the smallness condition of u and η, it implies

$$\frac{d}{dt} \mathcal{H}(t) \geq \widetilde{C}_1 \int \operatorname{sech}^2 \left(\frac{x}{\lambda} \right) u^2 + \widetilde{C}_2 \int \operatorname{sech}^2 \left(\frac{x}{\lambda} \right) (\partial_x u)^2$$
$$+ \widetilde{C}_3 \int \operatorname{sech}^2 \left(\frac{x}{\lambda} \right) \eta^2 + \widetilde{C}_4 \int \operatorname{sech}^2 \left(\frac{x}{\lambda} \right) (\partial_x \eta)^2, \tag{5.4}$$

for some $\widetilde{C}_j > 0$, $j = 1, 2, 3, 4$. Integrating the both sides of (5.4) in terms of t on $[2, \infty)$ with $\|u\|_{H^1} + \|\eta\|_{H^1} \lesssim \varepsilon$ exactly proves (5.1):

$$\int\limits_2^\infty \int \operatorname{sech}^2 \left(\frac{x}{\lambda} \right) \left(u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2 \right) (t, x) \, dx dt \lesssim \lambda \varepsilon^2.$$

The standard argument guarantees that there exists a sequence of time $t_n \to \infty$ as $n \to \infty$ (which can be chosen after passing to a subsequence) such that (5.2) holds true:

$$\lim_{n \to \infty} \int \operatorname{sech}^2 \left(\frac{x}{\lambda} \right) \left(u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2 \right) (t_n, x) \, dx = 0.$$

The proof is complete. □

5.1. Time dependent weights

With small modifications of the proofs established in Section 3, a stronger version of Proposition 5.1 can be obtained. Let $\lambda = \lambda(t)$ be the time-dependent weight given by

$$\lambda(t) := \frac{C_0 t}{\log^2 t}, \quad t \geq 2, \tag{5.5}$$

for any (fixed) constant $C_0 > 0$; precisely corresponding to the upper limit of the space interval $I(t)$ (1.9). The following result is stated for times $t \geq 2$, but it can be easily stated and proved for corresponding negative times.

Proposition 5.2 *(Decay in time-dependent intervals). Let (a, b, c) be dispersion-like parameters defined as in Definition 1.1. Let $(u, \eta)(t)$ be $H^1 \times H^1$ global solutions to (1.5) such that (1.16) holds. Then, we have*

$$\int_2^\infty \frac{1}{\lambda(t)} \int \operatorname{sech}^2\left(\frac{x}{\lambda(t)}\right) \left(u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2\right)(t, x) \, dx \, dt \lesssim \varepsilon^2. \tag{5.6}$$

As an immediate consequence, there exists an increasing sequence of time $\{t_n\}$ $(t_n \to \infty$ as $n \to \infty)$ such that

$$\int \operatorname{sech}^2\left(\frac{x}{\lambda(t_n)}\right) \left(u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2\right)(t_n, x) \, dx \longrightarrow 0 \text{ as } n \to \infty. \tag{5.7}$$

Proof. We choose, in functionals $\mathcal{I}, \mathcal{J}$ and $\mathcal{K}$ (see (3.1)–(3.3)), the time-dependent weight

$$\varphi(t, x) := \tanh\left(\frac{x}{\lambda(t)}\right) \quad \text{and} \quad \varphi'(t, x) := \frac{1}{\lambda(t)} \operatorname{sech}^2\left(\frac{x}{\lambda(t)}\right), \tag{5.8}$$

with $\lambda(t)$ given by (5.5). Note that

$$\frac{\lambda'(t)}{\lambda(t)} = \frac{1}{t}\left(1 - \frac{2}{\log t}\right). \tag{5.9}$$

Then, we compute (3.13), but in addition we will obtain the following three new terms:

$$-\frac{\lambda'(t)}{\lambda(t)} \int \frac{x}{\lambda(t)} \operatorname{sech}^2\left(\frac{x}{\lambda(t)}\right) (u\eta + \partial_x u \partial_x \eta), \tag{5.10}$$

$$-\alpha \frac{\lambda'(t)}{\lambda(t)} \int \left(1 - \frac{2x}{\lambda(t)} \tanh\left(\frac{x}{\lambda(t)}\right)\right) \frac{1}{\lambda(t)} \operatorname{sech}^2\left(\frac{x}{\lambda(t)}\right) \eta \partial_x u, \tag{5.11}$$

and

$$-\beta \frac{\lambda'(t)}{\lambda(t)} \int \left(1 - \frac{2x}{\lambda(t)} \tanh\left(\frac{x}{\lambda(t)}\right)\right) \frac{1}{\lambda(t)} \operatorname{sech}^2\left(\frac{x}{\lambda(t)}\right) u \partial_x \eta. \tag{5.12}$$

Let $c_0 := \frac{1}{2} \min(\widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3, \widetilde{C}_4)$, for $\widetilde{C}_i$ as in (5.4). In view of (5.4) (see also [42, Lemma 3.1] for complete details), we have the following estimate

$$|(5.10) + (5.11) + (5.12)|$$
$$\leq \frac{c_0}{\lambda(t)} \int \operatorname{sech}^2\left(\frac{x}{\lambda(t)}\right) \left(u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2\right) + \frac{\widetilde{C}\varepsilon^2}{t \log^2 t}, \tag{5.13}$$

for a fixed constant $\widetilde{C} > 0$. This enables us to obtain a new estimate for $\frac{d}{dt}\mathcal{H}(t)$ of the form

$$\frac{\widetilde{C}\varepsilon^2}{t\log^2 t} + \frac{d}{dt}\mathcal{H}(t) \geq \frac{\widetilde{C}_1}{2\lambda(t)}\int \operatorname{sech}^2\left(\frac{x}{\lambda(t)}\right)u^2 + \frac{\widetilde{C}_2}{2\lambda(t)}\int \operatorname{sech}^2\left(\frac{x}{\lambda(t)}\right)(\partial_x u)^2$$
$$+ \frac{\widetilde{C}_3}{2\lambda(t)}\int \operatorname{sech}^2\left(\frac{x}{\lambda(t)}\right)\eta^2 + \frac{\widetilde{C}_4}{2\lambda(t)}\int \operatorname{sech}^2\left(\frac{x}{\lambda(t)}\right)(\partial_x\eta)^2. \tag{5.14}$$

Then, the same argument as in the proof of Proposition 5.1 proves Proposition 5.2, since the left-hand side of (5.14) is integrable in t on $[2,\infty)$, but $\lambda(t)^{-1}$ does not integrate in $[2,\infty)$. $\quad\square$

6. Energy estimates

The proof of estimate (1.17) requires to show that the local H^1 norms of $(u,\eta)(t)$ converge to zero for all sequences $t_n \to +\infty$, and not only a particular one. In order to prove such a result, we will use an energy estimate, recalling that, for small solutions, the $H^1 \times H^1$ norm of $(u,\eta)(t)$ squared and the energy $E[u,\eta]$ in (1.4) are equivalent.

6.1. Preliminaries

Let $\psi = \psi(x)$ be a smooth, nonnegative and bounded function, to be chosen in the sequel. We consider the localized energy functional defined by

$$E_{loc}(t) = \frac{1}{2}\int \psi(x)\left(-a(\partial_x u)^2 - c(\partial_x\eta)^2 + u^2 + \eta^2 + u^2\eta\right)(t,x)dx. \tag{6.1}$$

(Compare with (1.4).) For the sake of simplicity, we introduce the following notations:

$$A := au_{xx} + u + u\eta, \qquad B := c\eta_{xx} + \eta + \frac{1}{2}u^2, \tag{6.2}$$
$$F := (1 - \partial_x^2)^{-1}A, \qquad G := (1 - \partial_x^2)^{-1}B.$$

(In other words, F and G are the canonical variables of A and B, respectively.) We will also use canonical variables f and g for u and η, and the following identities:

$$f_{xxx}g_x = (f_{xx}g_x)_x - f_{xx}g_{xx} \quad \text{and} \quad f_xg_{xxx} = (f_xg_{xx})_x - f_{xx}g_{xx}, \tag{6.3}$$

and

$$f_{xx}g = (f_xg)_x - f_xg_x \quad \text{and} \quad fg_{xx} = (fg_x)_x - f_xg_x. \tag{6.4}$$

6.2. Variation of local energy

We have

Lemma 6.1 *(Variation of local energy E_{loc}). Let u and η satisfy (1.5). Let f and g be canonical variables of u and η as in (4.1). Then, the following hold.*

1. *Time derivative. We have*

$$
\begin{aligned}
\frac{d}{dt} E_{loc}(t) = {}& \int \psi' f g + (1 - 2(a + c)) \int \psi' f_x g_x \\
& + (3ac - 2(a + c)) \int \psi' f_{xx} g_{xx} + 3ac \int \psi' f_{xxx} g_{xxx} \\
& - a \int \psi'' f_x g - c \int \psi'' f g_x \\
& + a(c - 2) \int \psi'' f_{xx} g_x + c(a - 2) \int \psi'' f_x g_{xx} \\
& + SNL_1(t) + SNL_2(t) + SNL_3(t) + SNL_4(t).
\end{aligned}
\tag{6.5}
$$

2. *The small nonlinear parts $SNL_j(t)$ are given by*

$$
SNL_1(t) := \frac{1}{2} a \int (\psi' u_x)_x (1 - \partial_x^2)^{-1}(u^2) + c \int (\psi' \eta_x)_x (1 - \partial_x^2)^{-1}(u\eta),
\tag{6.6}
$$

$$
\begin{aligned}
SNL_2(t) := {}& \frac{1}{2} \int \psi' f (1 - \partial_x^2)^{-1}(u^2) + \frac{a}{2} \int \psi' f_{xx} (1 - \partial_x^2)^{-1}(u^2) \\
& + \int \psi' g (1 - \partial_x^2)^{-1}(u\eta) + c \int \psi' g_{xx} (1 - \partial_x^2)^{-1}(u\eta) \\
& + \frac{1}{2} \int \psi' f_x (1 - \partial_x^2)^{-1}(u^2)_x + \int \psi' g_x (1 - \partial_x^2)^{-1}(u\eta)_x,
\end{aligned}
\tag{6.7}
$$

$$
SNL_3(t) := \frac{a}{2} \int \psi' f_{xxx} (1 - \partial_x^2)^{-1}(u^2)_x + c \int \psi' g_{xxx} (1 - \partial_x^2)^{-1}(u\eta)_x,
\tag{6.8}
$$

and

$$
\begin{aligned}
SNL_4(t) := {}& \frac{1}{2} \int \psi' (1 - \partial_x^2)^{-1}(u\eta)(1 - \partial_x^2)^{-1}(u^2) \\
& + \frac{1}{2} \int \psi' (1 - \partial_x^2)^{-1}(u\eta)_x (1 - \partial_x^2)^{-1}(u^2)_x.
\end{aligned}
\tag{6.9}
$$

Proof. We take the time derivative to $E_{loc}(t)$. The change of variable with notations A, B, F and G yields

$$
\begin{aligned}
\frac{d}{dt} E_{loc}(t) = {}& \int \psi \left(-a u_x u_{xt} + u u_t - c \eta_x \eta_{xt} + \eta \eta_t + u \eta u_t + \frac{1}{2} u^2 \eta_t \right) \\
= {}& \int \psi (u_t A + \eta_t B) + a \int \psi' u_t u_x + c \int \psi' \eta_t \eta_x \\
= {}& - \int \psi (G_x(F - F_{xx}) + F_x(G - G_{xx})) + a \int \psi' u_t u_x + c \int \psi' \eta_t \eta_x \\
= {}& \int \psi' F G - \int \psi' F_x G_x + a \int \psi' u_t u_x + c \int \psi' \eta_t \eta_x \\
=: {}& E_1 + E_2 + E_3 + E_4.
\end{aligned}
$$

We first focus on E_3 and E_4. Using (3.5) and the integration by parts gives

$$
\begin{aligned}
E_3 &= a \int \psi' u_t u_x \\
&= a \int \psi' u_x \left(c \partial_x \eta - (1+c)(1-\partial_x^2)^{-1} \partial_x \eta - (1-\partial_x^2)^{-1} \partial_x (\tfrac{1}{2} u^2) \right) \\
&= ac \int \psi' u_x \eta_x - a(c+1) \int \psi' u_x (1-\partial_x^2)^{-1} \eta_x \\
&\quad + \tfrac{1}{2} a \int (\psi' u_x)_x (1-\partial_x^2)^{-1} (u^2),
\end{aligned}
\tag{6.10}
$$

and

$$
\begin{aligned}
E_4 &= c \int \psi' \eta_t \eta_x \\
&= c \int \psi' \eta_x \left(a \partial_x u - (1+a)(1-\partial_x^2)^{-1} \partial_x u - (1-\partial_x^2)^{-1} \partial_x (u\eta) \right) \\
&= ac \int \psi' \eta_x u_x - c(a+1) \int \psi' \eta_x (1-\partial_x^2)^{-1} u_x \\
&\quad + c \int (\psi' \eta_x)_x (1-\partial_x^2)^{-1} (u\eta).
\end{aligned}
\tag{6.11}
$$

The last terms in both (6.10) and (6.11) corresponds to terms in $SNL_1(t)$. For the remaining terms in both (6.10) and (6.11), by using canonical variables f and g for u and η, we have

$$
\begin{aligned}
2ac \int \psi' u_x \eta_x &= 2ac \int \psi' (f_x g_x + f_{xxx} g_{xxx}) \\
&\quad - 2ac \int \psi' (f_{xxx} g_x + f_x g_{xxx}),
\end{aligned}
\tag{6.12}
$$

$$
\begin{aligned}
-a(c+1) \int \psi' u_x (1-\partial_x^2)^{-1} \eta_x &= -a(c+1) \int \psi' f_x g_x \\
&\quad + a(c+1) \int \psi' f_{xxx} g_x,
\end{aligned}
\tag{6.13}
$$

and

$$
\begin{aligned}
-c(a+1) \int \psi' \eta_x (1-\partial_x^2)^{-1} u_x &= -c(a+1) \int \psi' f_x g_x \\
&\quad + c(a+1) \int \psi' f_x g_{xxx}.
\end{aligned}
\tag{6.14}
$$

Substituting the identities (6.3) into (6.12)–(6.14), and collecting all terms we have

$$
\begin{aligned}
E_3 + E_4 &= -(a+c) \int \psi' f_x g_x + (2ac - (a+c)) \int \psi' f_{xx} g_{xx} \\
&\quad + 2ac \int \psi' f_{xxx} g_{xxx} \\
&\quad + a(c-1) \int \psi'' f_{xx} g_x + c(a-1) \int \psi'' f_x g_{xx} \\
&\quad + SNL_1(t).
\end{aligned}
\tag{6.15}
$$

Now we deal with E_1 and E_2. From (6.2) we have

$$F = a\, f_{xx} + f + (1 - \partial_x^2)^{-1}(u\eta) \quad \text{and} \quad G = c\, g_{xx} + g + \frac{1}{2}(1 - \partial_x^2)^{-1}(u^2).$$

Using (6.3) and (6.4), then a direct calculation gives

$$\begin{aligned}
E_1 = \int \psi' FG &= ac \int \psi' f_{xx} g_{xx} - (a+c) \int \psi' f_x g_x + \int \psi' fg \\
&\quad - a \int \psi'' f_x g - c \int \psi'' f g_x \\
&\quad + \frac{a}{2} \int \psi' f_{xx} (1 - \partial_x^2)^{-1}(u^2) + \frac{1}{2} \int \psi' f (1 - \partial_x^2)^{-1}(u^2) \\
&\quad + c \int \psi' g_{xx} (1 - \partial_x^2)^{-1}(u\eta) + \int \psi' g (1 - \partial_x^2)^{-1}(u\eta) \\
&\quad + \frac{1}{2} \int \psi' (1 - \partial_x^2)^{-1}(u\eta)(1 - \partial_x^2)^{-1}(u^2),
\end{aligned} \tag{6.16}$$

and

$$\begin{aligned}
E_2 = \int \psi' F_x G_x &= ac \int \psi' f_{xxx} g_{xxx} - (a+c) \int \psi' f_{xx} g_{xx} + \int \psi' f_x g_x \\
&\quad - a \int \psi'' f_{xx} g_x - c \int \psi'' f_x g_{xx} \\
&\quad + \frac{a}{2} \int \psi' f_{xxx} (1 - \partial_x^2)^{-1}(u^2)_x + \frac{1}{2} \int \psi' f_x (1 - \partial_x^2)^{-1}(u^2)_x \\
&\quad + c \int \psi' g_{xxx} (1 - \partial_x^2)^{-1}(u\eta)_x + \int \psi' g_x (1 - \partial_x^2)^{-1}(u\eta)_x \\
&\quad + \frac{1}{2} \int \psi' (1 - \partial_x^2)^{-1}(u\eta)_x (1 - \partial_x^2)^{-1}(u^2)_x,
\end{aligned} \tag{6.17}$$

respectively. (6.15), (6.16) and (6.17) prove (6.5) with (6.6)–(6.9). □

Remark 6.1. The previous computations can be easily extended to the case of a weight $\psi(x)$ in (6.1) depending on time, after some direct estimates on the new emergent terms are carried out. We will prove and use this fact in next Section.

7. Proof of the Theorem 1.4

Now we finally prove Theorem 1.4. For the sake of simplicity in some computations, we first deal with the case of a time independent parameter λ, and then we extend the result to the case of $\lambda(t)$ given in (5.5). Let us take

$$\psi(x) := \lambda \operatorname{sech}^4\left(\frac{x}{\lambda}\right) = \lambda(\varphi')^2, \tag{7.1}$$

in the localized energy $E_{loc}(t)$ (6.1), where φ' is chosen as in the proof of Proposition 5.1. Note that

$$|\psi'(x)| \le 4\varphi' \quad \text{and} \quad |\psi''(x)| \le \frac{20}{\lambda}\varphi'. \tag{7.2}$$

Using Proposition 5.1 and Lemmas 6.1, 2.5, 2.6, 2.7, 2.2 and 2.3 carries out the following Proposition:

Proposition 7.1. *Let (a, b, c) be* dispersion-like *parameters defined as in Definition 1.1. Let $(u, \eta)(t)$ be $H^1 \times H^1$ global solutions to (1.5) such that (1.16) holds. Then, we have for large (but fixed) $\lambda \gg 1$ that*

$$\lim_{t \to \infty} \int \mathrm{sech}^4 \left(\frac{x}{\lambda} \right) \left(u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2 \right) (t, x) \, dx = 0. \tag{7.3}$$

Proof. The Cauchy–Schwarz inequality and (7.2) allows that first four lines in the right-hand side of (6.5) are bounded by the terms

$$\int \varphi'(a_1 f^2 + a_2 f_x^2 + a_3 f_{xx}^2 + a_4 f_{xxx}^2)$$

$$+ \int \varphi'(b_1 g^2 + b_2 g_x^2 + b_3 g_{xx}^2 + b_4 g_{xxx}^2), \tag{7.4}$$

for some positive constants a_j, b_j, $j = 1, 2, 3, 4$. Coming back to standard variables, Lemmas 2.2 and 2.3 guarantee that

$$(7.4) \lesssim \int \varphi'(u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2). \tag{7.5}$$

It remains to control the nonlinear terms $SNL_i(t)$, $i = 1, 2, 3, 4$, in (6.5). For $SNL_1(t)$, we use Lemma 2.6 to obtain

$$|SNL_1(t)| \lesssim (\|u\|_{H^1} + \|\eta\|_{H^1}) \int \varphi'(u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2). \tag{7.6}$$

For $SNL_2(t)$, since f and g are in H^3 (with $\|f\|_{H^3} \lesssim \|u\|_{H^1}$ and $\|f\|_{H^3} \lesssim \|u\|_{H^1}$), we have from Lemma 2.5 that

$$|SNL_2(t)| \lesssim (\|u\|_{H^1} + \|\eta\|_{H^1}) \int \varphi'(u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2). \tag{7.7}$$

For $SNL_3(t)$, we use the Sobolev embedding ($\|f_{xx}\|_{L^\infty} \lesssim \|u\|_{H^1}$) and Lemma 2.7 to obtain

$$|SNL_3(t)| \lesssim (\|u\|_{H^1} + \|\eta\|_{H^1}) \int \varphi'(u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2). \tag{7.8}$$

Lastly, for $SNL_4(t)$, note that

$$\left\| (1 - \partial_x^2)^{-1}(u\eta) \right\|_{L^\infty} + \left\| (1 - \partial_x^2)^{-1}(u\eta)_x \right\|_{L^\infty} \lesssim \|u\eta\|_{L^2} \lesssim (\|u\|_{H^1} + \|\eta\|_{H^1}).$$

Lemma 2.5 gives

$$|SNL_4(t)| \lesssim (\|u\|_{H^1} + \|\eta\|_{H^1}) \int \varphi'(u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2). \tag{7.9}$$

Collecting all (7.5)–(7.9) we obtain

$$\left| \frac{d}{dt} E_{loc}(t) \right| \lesssim \int \mathrm{sech}^2 \left(\frac{x}{\lambda} \right) \left(u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2 \right) (t, x) \, dx. \tag{7.10}$$

Integrating on $[t, t_n]$, for $t < t_n$ as in (5.2) and Proposition 5.1 yield

$$|E_{loc}(t_n) - E_{loc}(t)| \lesssim \int_t^\infty \int \operatorname{sech}^2 \left(\frac{x}{\lambda} \right) (u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2)(t, x) \, dx dt$$

$$< \infty.$$

Note that, from the Sobolev embedding (with $\|\eta\|_{H^1} \lesssim \varepsilon$), we have

$$|E_{loc}(t_n)| \lesssim \int \varphi'(u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2)(t_n) \longrightarrow 0, \quad \text{as } n \to \infty,$$

thanks to Proposition 5.1. Thus, by sending $t_n \to \infty$, we have

$$\int \operatorname{sech}^4 \left(\frac{x}{\lambda} \right) (u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2 + \eta u^2)(t)$$

$$\lesssim \int_t^\infty \int \operatorname{sech}^2 \left(\frac{x}{\lambda} \right) (u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2).$$

Once again, the Sobolev embedding (with $\|\eta\|_{H^1} \lesssim \varepsilon$) guarantees that

$$\lim_{t \to \infty} \int \operatorname{sech}^4 \left(\frac{x}{\lambda} \right) (u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2)(t) = 0,$$

which completes the proof of Proposition 7.1. $\square$

Replacing the parameter λ by the time-dependent function $\lambda(t)$ defined in (5.5), Proposition 7.1 extends to the case of a time-dependent, increasing interval in space. First, we take in (6.5):

$$\psi(t, x) := \operatorname{sech}^4 \left(\frac{x}{\lambda(t)} \right). \tag{7.11}$$

Proposition 7.2. *Let (a, b, c) be* dispersion-like *parameters defined as in Definition 1.1. Let $(u, \eta)(t)$ be $H^1 \times H^1$ global solutions to (1.5) such that (1.16) holds. Then, we have*

$$\lim_{t \to \infty} \int \operatorname{sech}^4 \left(\frac{x}{\lambda(t)} \right) \left(u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2 \right) (t, x) \, dx = 0. \tag{7.12}$$

Remark 7.1. This last proposition finally proves Theorem 1.4, since

$$\int \operatorname{sech}^4 \left(\frac{x}{\lambda(t)} \right) \left(u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2 \right) (t, x) \, dx$$

$$\gtrsim \int_{-\lambda(t)}^{\lambda(t)} \left(u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2 \right) (t, x) \, dx.$$

Proof. Similarly as the proof of Proposition 5.2, it suffices to deal with the following additional term in the energy estimate (6.5):

$$\int \psi_t \left(u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2 \right). \tag{7.13}$$

Note that from (5.9) and (7.11),

$$|\psi_t| = \left| 4\frac{\lambda'(t)}{\lambda(t)}\frac{x}{\lambda(t)}\operatorname{sech}^4\left(\frac{x}{\lambda(t)}\right)\tanh\left(\frac{x}{\lambda(t)}\right)\right| \lesssim \frac{1}{\lambda(t)}\operatorname{sech}^2\left(\frac{x}{\lambda(t)}\right).$$

This estimate immediately implies

$$(7.13) \lesssim \frac{1}{\lambda(t)}\int \operatorname{sech}^2\left(\frac{x}{\lambda(t)}\right)(u^2 + (\partial_x u)^2 + \eta^2 + (\partial_x \eta)^2). \tag{7.14}$$

Therefore, Proposition 5.2, and the same argument of the proof of Proposition 7.1, together with estimate (7.14) prove Proposition 7.2. □

Compliance with ethical standards

The authors of this article certify that:

1. **Disclosure of potential conflicts of interest.** No potential conflicts of interest were detected.
2. **Research involving Human Participants and/or Animals.** No research of this type was done.
3. **Informed consent.** No informed consent applies.

We certify the originality of the research involved in this manuscript.

Appendix A. Full description of Remark 1.7

First of all, we recall the data given in Remark 1.7. The dispersion relation is given by

$$w(k) = \frac{\pm|k|}{1+k^2}(1 - ak^2)^{1/2}(1 - ck^2)^{1/2},$$

and the group velocity is

$$|w'(k)| = \frac{|ack^6 + 3ack^4 - (1 + 2a + 2c)k^2 + 1|}{(1+k^2)^2(1-ak^2)^{1/2}(1-ck^2)^{1/2}}.$$

Since $a + c = -2 + \frac{1}{3b}$, we get

$$ack^6 + 3ack^4 - (1 + 2a + 2c)k^2 + 1 = ack^6 + 3ack^4 + \left(3 - \frac{2}{3b}\right)k^2 + 1.$$

The cubic polynomial $p(\mu) := ac\mu^3 + 3ac\mu^2 + (3 - \frac{2}{3b})\mu + 1$ ($\mu \geq 0$, $ac > 0$) never vanishes if its minimum value is positive. The derivative $p'(\mu) = 3ac\mu^2 + 6ac\mu + (3 - \frac{2}{3b})$ has roots $\mu_\pm = -1 \pm (1 + \frac{2/(3b)-3}{3ac})^{1/2}$, for which the condition $\mu_\pm > 0$ does not hold if $b \geq 2/9$. In this case readily we have $p(\mu) \geq 1$ for all $\mu \geq 0$. In consequence, $|w'(k)| > 0$ for all $k \in \mathbb{R}$.

Appendix B. Alternative expression of $\mathcal{P}$ in (4.26)

We recall from Subsection 2.1 the alternative expression (2.5) of the set $\mathcal{P}_0(b) = \mathcal{B}_0 \cap \mathcal{B}_1(b) \cap \mathcal{B}_2(b) \cap \mathcal{B}_3(b)$, where

$$\mathcal{P}_0(b) = \left\{(a, b, c) = \left(-\frac{\nu}{2} + \frac{1}{3} - b, \, b, \, \frac{\nu}{2} - b\right) : \nu \in [0, 1] \cap \left(\frac{2}{3} - 2b, 2b\right)\right\} \neq \emptyset, \tag{B.1}$$

for $b > 1/6$. Note in addition that the open interval

$$\left(\frac{2}{3} - 2b, 2b\right) \neq \emptyset$$

for $b > 1/6$. Recall the set $\mathcal{P}_0$ defined in (2.6):

$$\mathcal{P}_0 = \bigcup_{b > 1/6} \mathcal{P}_0(b).$$

Note that $\mathcal{P}_0$ is precisely the image (a, b, c) of the bold set in (b, ν) in Fig. 1, right panel. Then, Lemma 2.1 can be rewritten as

Lemma B.1. *There exist (a, b, c, d) such that* (1.2) *and* (1.3) *are satisfied if and only if $b > \frac{1}{6}$. In this case, $(a, b, c) \in \mathcal{P}_0$.*

This is the starting point. Recall again the set $\mathcal{B}_4(b)$ in Subsection 4.2 explaining *dispersion-like* parameters:

$$\mathcal{B}_4(b) := \left\{ (a, c) \in \mathbb{R}^2 \ : \ 3b(a + c) + 2b^2 - 8ac < 0 \right\}.$$

According to (B.1), let us substitute

$$a = -\frac{\nu}{2} + \frac{1}{3} - b \qquad \text{and} \qquad c = \frac{\nu}{2} - b$$

into $3b(a + c) + 2b^2 - 8ac < 0$. We get the hyperbola-like expression in (ν, b)

$$6 \left(b - \frac{11}{72} \right)^2 - \left(\nu - \frac{1}{3} \right)^2 > \frac{25}{864}. \tag{B.2}$$

Hence, $\mathcal{P}_0(b) \cap \mathcal{B}_4(b) = \widetilde{\mathcal{B}}_4(b)$, where

$$\widetilde{\mathcal{B}}_4(b) := \left\{ \left(-\frac{\nu}{2} + \frac{1}{3} - b, \ b, \ \frac{\nu}{2} - b \right) \ : \ 6 \left(b - \frac{11}{72} \right)^2 - \left(\nu - \frac{1}{3} \right)^2 > \frac{25}{864}, \right.$$
$$\left. \nu \in [0, 1] \cap \left(\frac{2}{3} - 2b, 2b \right) \right\}.$$

Let us describe in more detail $\widetilde{\mathcal{B}}_4(b)$. For that, we need to consider the following two simultaneous systems of inequalities

$$\begin{cases} \nu > \frac{2}{3} - 2b \\ 6 \left(b - \frac{11}{72} \right)^2 - \left(\nu - \frac{1}{3} \right)^2 > \frac{25}{864}, \end{cases} \tag{B.3}$$

and

$$\begin{cases} \nu < 2b \\ 6 \left(b - \frac{11}{72} \right)^2 - \left(\nu - \frac{1}{3} \right)^2 > \frac{25}{864}. \end{cases} \tag{B.4}$$

For (B.3) and $b > 0$, we have solutions

$$\frac{1}{3} - \sqrt{6b^2 - \frac{11}{6}b + \frac{1}{9}} < \nu, \qquad b < \frac{1}{4}, \tag{B.5}$$

and

$$\frac{2}{3} - 2b < \nu, \qquad b \geq \frac{1}{4}. \tag{B.6}$$

On the other hand, for (B.4),

$$\nu < \frac{1}{3} + \sqrt{6b^2 - \frac{11}{6}b + \frac{1}{9}}, \qquad b < \frac{1}{4}, \tag{B.7}$$

and

$$\nu < 2b, \qquad b \geq \frac{1}{4}. \tag{B.8}$$

Collecting (B.5)–(B.8), we get

$$\frac{1}{3} - \sqrt{6b^2 - \frac{11}{6}b + \frac{1}{9}} < \nu < \frac{1}{3} + \sqrt{6b^2 - \frac{11}{6}b + \frac{1}{9}}, \qquad b < \frac{1}{4}, \tag{B.9}$$

and

$$\frac{2}{3} - b < \nu < 2b, \qquad b \geq \frac{1}{4}. \tag{B.10}$$

Therefore, from these observations, we know that $\widetilde{\mathcal{B}}_4 \neq \emptyset$ provided $b > \frac{2}{9}$. Indeed, this is because the interval

$$\left(\frac{1}{3} - \sqrt{6b^2 - \frac{11}{6}b + \frac{1}{9}}, \ \frac{1}{3} + \sqrt{6b^2 - \frac{11}{6}b + \frac{1}{9}} \right)$$

is nonempty only for $b > \frac{2}{9}$. Moreover, if now

$$I_b := [0, 1] \cap \left(\frac{2}{3} - 2b, \ 2b \right) \cap \left(\frac{1}{3} - \sqrt{6b^2 - \frac{11}{6}b + \frac{1}{9}}, \ \frac{1}{3} + \sqrt{6b^2 - \frac{11}{6}b + \frac{1}{9}} \right),$$

then we have

$$I_b = \begin{cases} \left(\frac{1}{3} - \sqrt{6b^2 - \frac{11}{6}b + \frac{1}{9}}, \frac{1}{3} + \sqrt{6b^2 - \frac{11}{6}b + \frac{1}{9}} \right), & \frac{2}{9} < b \leq \frac{1}{4}, \\ \left(\frac{2}{3} - 2b, 2b \right), & \frac{1}{4} < b \leq \frac{1}{3}, \\ [0, 2b), & \frac{1}{3} < b \leq \frac{1}{2}, \\ [0, 1], & \frac{1}{2} < b. \end{cases}$$

Concluding, we can define

$$\mathcal{P}(b) := \widetilde{\mathcal{B}}_4(b) = \left\{ \left(-\frac{\nu}{2} + \frac{1}{3} - b, \ b, \ \frac{\nu}{2} - b \right) : \nu \in I_b \right\},$$

for $b > \frac{2}{9}$. Let

$$\mathcal{P} := \bigcup_{b > \frac{2}{9}} \mathcal{P}(b).$$

Then $\mathcal{P}$ describes all parameters (a, b, c) for which Theorem 1.4 holds.

References

[1] M.A. Alejo, Nonlinear stability of Gardner breathers, J. Differ. Equ. 264 (2) (2018) 1192–1230.
[2] M.A. Alejo, C. Muñoz, Nonlinear stability of mKdV breathers, Commun. Math. Phys. 324 (1) (2013) 233–262.
[3] M.A. Alejo, C. Muñoz, Dynamics of complex-valued modified KdV solitons with applications to the stability of breathers, Anal. PDE 8 (3) (2015) 629–674.
[4] M.A. Alejo, C. Muñoz, Almost sharp nonlinear scattering in one-dimensional Born–Infeld equations arising in nonlinear electrodynamics, Proc. Am. Math. Soc. 146 (5) (2018) 2225–2237, https://doi.org/10.1090/proc/13947.
[5] M.A. Alejo, C. Muñoz, J.M. Palacios, On the variational structure of breather solutions I: Sine-Gordon equation, J. Math. Anal. Appl. 453 (2) (2017) 1111–1138.
[6] M.A. Alejo, C. Muñoz, J.M. Palacios, On the variational structure of breather solutions II: periodic mKdV equation, Electron. J. Differ. Equ. (2017) 56.
[7] B. Alvarez-Samaniego, D. Lannes, Large time existence for 3D water-waves and asymptotics, Invent. Math. 171 (3) (2008) 485–541.
[8] Charles J. Amick, Regularity and uniqueness of solutions to the Boussinesq system of equations, J. Differ. Equ. 54 (2) (1984) 231–247.
[9] Ellen ShiTing Bao, Robin Ming Chen, Qing Liu, Existence and symmetry of ground states to the Boussinesq abcd systems, Arch. Ration. Mech. Anal. 216 (2) (May 2015) 569–591.
[10] T.B. Benjamin, J.L. Bona, J.J. Mahony, Model equations for long waves in nonlinear dispersive systems, Philos. Trans. R. Soc. Lond. Ser. A, Math. Phys. Sci. 272 (1220) (1972) 47–78.
[11] J.L. Bona, M. Chen, J.-C. Saut, Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I: derivation and linear theory, J. Nonlinear Sci. 12 (2002) 283–318.
[12] J.L. Bona, M. Chen, J.-C. Saut, Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II: the nonlinear theory, Nonlinearity 17 (2004) 925–952.
[13] J.L. Bona, T. Colin, D. Lannes, Long wave approximations for water waves, Arch. Ration. Mech. Anal. 178 (3) (2005) 373–410.
[14] J.L. Bona, D. Lannes, J.-C. Saut, Asymptotic models for internal waves, J. Math. Pures Appl. (9) 89 (6) (2008) 538–566.
[15] J.L. Bona, R.L. Sachs, Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation, Commun. Math. Phys. 118 (1) (1988) 15–29.
[16] J.L. Bona, N. Tzvetkov, Sharp well-posedness results for the BBM equation, Discrete Contin. Dyn. Syst. 23 (4) (2009) 1241–1252.
[17] J. Boussinesq, Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond, J. Math. Pures Appl. (2) 17 (1872) 55–108.
[18] C. Burtea, New long time existence results for a class of Boussinesq-type systems, J. Math. Pures Appl. (9) 106 (2) (2016) 203–236.
[19] Min Chen, Nghiem V. Nguyen, Shu-Ming Sun, Existence of traveling-wave solutions to Boussinesq systems, Differ. Integral Equ. 24 (9–10) (2011) 895–908.
[20] Min Chen, Nghiem V. Nguyen, Shu-Ming Sun, Solitary-wave solutions to Boussinesq systems with large surface tension, Discrete Contin. Dyn. Syst. 26 (4) (2010) 1153–1184.
[21] R.M. Chen, Yue Liu, On the ill-posedness of a weakly dispersive one-dimensional Boussinesq system, J. Anal. Math. 121 (2013) 299–316.
[22] Yonggeun Cho, Tohru Ozawa, On small amplitude solutions to the generalized Boussinesq equations, Discrete Contin. Dyn. Syst. 17 (4) (2007) 691–711.
[23] Raphaël Côte, Claudio Muñoz, Didier Pilod, Gideon Simpson, Asymptotic stability of high-dimensional Zakharov–Kuznetsov solitons, Arch. Ration. Mech. Anal. 220 (2) (2016) 639–710.
[24] K. El Dika, Smoothing effect of the generalized BBM equation for localized solutions moving to the right, Discrete Contin. Dyn. Syst. 12 (5) (2005) 973–982.
[25] K. El Dika, Y. Martel, Stability of N solitary waves for the generalized BBM equations, Dyn. Partial Differ. Equ. 1 (4) (2004) 401–437.
[26] S. Hakkaev, M. Stanislavova, A. Stefanov, Spectral stability for subsonic traveling pulses of the Boussinesq "abc" system, SIAM J. Appl. Dyn. Syst. 12 (2) (2013) 878–898.
[27] M. Kowalczyk, Y. Martel, C. Muñoz, Kink dynamics in the ϕ^4 model: asymptotic stability for odd perturbations in the energy space, J. Am. Math. Soc. 30 (2017) 769–798.
[28] M. Kowalczyk, Y. Martel, C. Muñoz, Nonexistence of small, odd breathers for a class of nonlinear wave equations, Lett. Math. Phys. 107 (5) (May 2017) 921–931.
[29] M. Kowalczyk, Y. Martel, C. Muñoz, On asymptotic stability of nonlinear waves, Laurent Schwartz seminar notes (2017), see url at http://slsedp.cedram.org/slsedp-bin/fitem?id=SLSEDP_2016-2017_____A18_0.

[30] C. Kwak, C. Muñoz, Extended decay estimates for generalized BBM equations, preprint arXiv:1802.01925.

[31] F. Linares, Global existence of small solutions for a generalized Boussinesq equation, J. Differ. Equ. 106 (1993) 257–293.

[32] F. Linares, M. Scialom, Asymptotic behavior of solutions of a generalized Boussinesq type equation, Nonlinear Anal. 25 (11) (1995) 1147–1158.

[33] F. Linares, D. Pilod, J.-C. Saut, Well-posedness of strongly dispersive two-dimensional surface wave Boussinesq systems, SIAM J. Math. Anal. 44 (6) (2012) 4195–4221.

[34] H. Lindbland, T. Tao, Asymptotic decay for a one-dimensional nonlinear wave equation, Anal. PDE 5 (2) (2012) 411–422, https://doi.org/10.2140/apde.2012.5.411.

[35] Yue Liu, Instability of solitary waves for generalized Boussinesq equations, J. Dyn. Differ. Equ. 53 (1993) 537–558.

[36] Yue Liu, Decay and scattering of small solutions of a generalized Boussinesq equation, J. Funct. Anal. 147 (1997) 51–68.

[37] Y. Martel, F. Merle, A Liouville theorem for the critical generalized Korteweg–de Vries equation, J. Math. Pures Appl. (9) 79 (4) (2000) 339–425.

[38] Y. Martel, F. Merle, Asymptotic stability of solitons for subcritical generalized KdV equations, Arch. Ration. Mech. Anal. 157 (3) (2001) 219–254.

[39] Y. Martel, F. Merle, Asymptotic stability of solitons for subcritical gKdV equations revisited, Nonlinearity 18 (1) (2005) 55–80.

[40] F. Merle, P. Raphaël, The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation, Ann. Math. (2) 161 (1) (2005) 157–222.

[41] C. Muñoz, Instability in nonlinear Schrödinger breathers, Proyecciones 36 (4) (2017) 653–683.

[42] C. Muñoz, F. Poblete, J.C. Pozo, Scattering in the energy space for Boussinesq equations, Commun. Math. Phys. 361 (1) (2018) 127–141, https://doi.org/10.1007/s00220-018-3099-7; preprint arXiv:1707.02616, 2017.

[43] J.C. Muñoz, Y. Rivas, Existence and asymptotic decay for a Boussinesq-type model, J. Math. Anal. Appl. 345 (2008) 455–475.

[44] Filipe Oliveira, A note on the existence of traveling-wave solutions to a Boussinesq system, Differ. Integral Equ. 29 (1–2) (2016) 127–136.

[45] J.C. Saut, Asymptotic models for surface and internal waves, in: 29th Coloquio Brasileiro de Matemática, Publicações matemáticas IMPA, 2013.

[46] J.-C. Saut, L. Xu, The Cauchy problem on large time for surface waves Boussinesq systems, J. Math. Pures Appl. (9) 97 (6) (2012) 635–662.

[47] J.-C. Saut, C. Wang, L. Xu, The Cauchy problem on large time for surface-waves-type Boussinesq systems II, SIAM J. Math. Anal. 49 (4) (2017) 2321–2386.

[48] Maria E. Schonbek, Existence of solutions for the Boussinesq system of equations, J. Differ. Equ. 42 (3) (1981) 325–352.

[49] S. Snelson, Asymptotic stability for odd perturbations of the stationary kink in the variable-speed ϕ^4 model, Trans. Am. Math. Soc. (2018), https://doi.org/10.1090/tran/7300, in press.