Analysis and Mathematical Physics (2018) 8:655–677
https://doi.org/10.1007/s13324-018-0253-5

Uniformizations of stable (γ, n)-gonal Riemann surfaces

Rubén A. Hidalgo[1]

Received: 14 February 2018 / Revised: 25 September 2018 / Accepted: 5 October 2018 /
Published online: 15 October 2018

Abstract

A (γ, n)-gonal pair is a pair (S, f), where S is a closed Riemann surface and $f : S \to R$ is a degree n holomorphic map onto a closed Riemann surface R of genus γ. If the signature of (S, f) is of hyperbolic type, then it admits a uniformizing pair (Γ, G), where G is a Fuchsian group acting on the unit disc $\mathbb{D}$ containing Γ as an index n subgroup, such that f is induced by the inclusion $\Gamma \leq G$. The uniformizing pair is uniquely determined by (S, f), up to conjugation by holomorphic automorphisms of $\mathbb{D}$, and it permits to provide a natural complex orbifold structure on the Hurwitz space parametrizing (twisted) isomorphic classes of pairs topologically equivalent to (S, f). In order to produce certain compactifications of these Hurwitz spaces, one needs to consider the so called stable (γ, n)-gonal pairs, which are natural geometrical deformations of (γ, n)-gonal pairs. Due to the above, it seems interesting to search for uniformizations of stable (γ, n)-gonal pairs, in terms of certain class of Kleinian groups. In this paper we review such uniformizations by using noded Fuchsian groups, obtained from the noded Beltrami differentials of Fuchsian groups that were previously studied by Alexander Vasil'ev and the author, and which provide uniformizations of stable Riemann orbifolds. These uniformizations permit to obtain a compactification of the Hurwitz spaces together a complex orbifold structure, these being quotients of the augmented Teichmüller space of G by a suitable finite index subgroup of its modular group.

Keywords Riemann surfaces · Fuchsian groups · Teichmüller space · Quasiconformal maps

Mathematics Subject Classification Primary 30F10 · 30F35; Secondary 30F40 · 30F60

To the memory of Alexander Vasil'ev.

Supported by Project Fondecyt 1150003 and Project Anillo ACT1415 PIA CONICYT.

✉ Rubén A. Hidalgo
ruben.hidalgo@ufrontera.cl

[1] Departamento de Matemática y Estadística, Universidad de La Frontera, Temuco, Chile

1 Introduction

Let $\gamma \geq 0$ and $n \geq 2$ be integers. A pair (S, f) is called a (γ, n)-*gonal pair* if S is a closed Riemann surface and $f : S \to R$ is a degree n holomorphic branched cover onto a closed Riemann surface R of genus γ; in this case, we say that S is (γ, n)-*gonal*. A $(0, n)$-gonal pair is usually called an n-*gonal pair*. In the case that the branched cover f is *simple*, that is, each branch value of f has exactly $n-1$ preimages, we talk of a *simple* (γ, n)-*gonal pair* (a simple $(0, n)$-gonal pair is just called *simple* n-*gonal*). There are different notions of equivalences between (γ, n)-gonal pairs and we are interested in three of them. If $(S_1, f_1 : S_1 \to R_1)$ and $(S_2, f_2 : S_2 \to R_2)$ are two (γ, n)-gonal pairs, then we say that they are:

(i) *topologically equivalent* if there are orientation-preserving homeomorphisms $\phi : S_1 \to S_2$ and $\psi : R_1 \to R_2$ such that $f_2 \circ \phi = \psi \circ f_1$,
(ii) *twisted isomorphic* if in (i) we may assume ϕ and ψ to be isomorphisms, and
(iii) *isomorphic* if in (i) we assume $R_1 = R_2$, ϕ an isomorphism and ψ the identity map.

Associated to each (γ, n)-gonal pair (S, f) is its Hurwitz space $\mathcal{H}_0(S, f)$ (respectively, $\mathcal{H}(S, f)$) consisting of the isomorphic classes (respectively, the twisted isomorphic classes) of (γ, n)-gonal pairs topologically equivalent to (S, f). If (S', f') is topologically equivalent to (S, f), then there are natural bijections between $\mathcal{H}_0(S, f)$ and $\mathcal{H}_0(S', f')$ (respectively, between $\mathcal{H}(S, f)$ and $\mathcal{H}(S', f')$).

In 1873, Clebsch [16] proved that any two simple n-gonal pairs are topologically equivalent, in particular, for a simple n-gonal pair (S, f), the space $\mathcal{H}(S, f)$ is the classical Hurwitz space parametrizing twisted isomorphic classes of degree n simple covers of $\widehat{\mathbb{C}}$. Under the previous assumption, in 1891, Hurwitz [34] proved that $\mathcal{H}(S, f)$ has the structure of a complex manifold of dimension $r-3$, where S has genus g and $r = 2g + 2n - 2 \geq 3$; so, by Clebsch's result, $\mathcal{H}(S, f)$ is irreducible. As every Riemann surface of genus g admits a degree $n \geq g+1$ simple branched cover over $\widehat{\mathbb{C}}$ (by Riemann–Roch's theorem), Severi [54] used the above facts to prove that the moduli space $\mathcal{M}_g$, of closed Riemann surfaces of genus g, is irreducible. For the case of n-gonal pairs (S, f), not necessarily simple ones, Fried [26] proved that $\mathcal{H}(S, f)$ also has the structure of a connected complex manifold of dimension $r-3$.

Let $(S, f : S \to \widehat{\mathbb{C}})$ be an n-gonal pair (not necessarily a simple one) whose set B_f of branch values has cardinality $|B_f| \geq 3$. In [50,51], Natanzon constructed a pair (Γ, G), where G is a Fuchsian group, isomorphic to a free group of rank $|B_f|-1$, and a suitable finite index subgroup Γ of G (of index n) such that the holomorphic cover map $f : S - f^{-1}(B_f) \to \widehat{\mathbb{C}} - B_f$ is induced by the inclusion of Γ in G. Such a pair is called an uniformizing pair of (S, f). By using these uniformizing pairs, he was able to observe that the Hurwitz space $\mathcal{H}(S, f)$ is homeomorphic to the a quotient $\mathbb{R}^m/M$, where $m = 2(|B_f|-3)$ and M is a certain discrete group of orientation-preserving homeomorphisms of $\mathbb{R}^m$ (in this case, $\mathbb{R}^m$ is the real structure of the Teichmüller space of the Riemann sphere punctured at $|B_f|$ points).

In Sect. 5, for a general (γ, n)-gonal pair (S, f), we define the notion for it to be of *hyperbolic type* and, in such a case, we associate to it a pair (Γ, G), called a *uniformizing pair* of (S, f), where G is a Fuchsian group acting on the unit disc $\mathbb{D}$

containing Γ as an index n subgroup, such that there are isomorphisms $\phi : S \to \mathbb{D}/\Gamma$ and $\psi : R \to \mathbb{D}/G$ with $\pi = \psi f \phi^{-1}$ being a branched covering induced by the inclusion of Γ inside G. The uniformizing pair is uniquely determined up to conjugation by holomorphic automorphisms of $\mathbb{D}$. Using such an uniformization pair, it is possible to obtain the following fact that generalizes the previous Natanzon's result.

Theorem 1 *Let (S, f) be a (γ, n)-gonal pair of hyperbolic type and let (Γ, G) be a unifomizing pair of it. Then*

(i) $\mathcal{H}_0(S, f)$ is isomorphic to the Teichmüller space $T(\mathbb{D}, G)$ of the Riemann orbifold $\mathbb{D}/G$, this being a finite dimensional simply-connected complex manifold, and
(ii) $\mathcal{H}(S, f)$ is a complex orbifold, isomorphic to a quotient $T(\mathbb{D}, G)/M(\Gamma, G)$, where $M(\Gamma, G)$ is a suitable finite index subgroup of the group of holomorphic automorphisms of $T(\mathbb{D}, G)$.

In [8,15,27] there is some study of (γ, n)-gonal pairs, and their groups of automorphisms, when n is a prime integer and the n-gonal map is a regular branched cover.

In the literature, compactifications of Hurwitz spaces of simple n-gonal pairs has been obtained (see for instance [22,25,28]) by adding some degenerations, called stable n-gonal pairs (also called admissible ones by some authors), which are certain geometric degenerations of simple n-gonal pairs. Other examples of compactifications of Hurwitz spaces of dimension one have been obtained in [18,21]. Next, we proceed to recall such kind of degenerations in the more general context (i.e., it might be that either $\gamma \geq 0$ and/or that f is non-simple). Similar constructions already have been done in previous papers, such as [19,20,23].

Let $(S, f : S \to R)$ be a fixed (γ, n)-gonal pair of hyperbolic type (see Sect. 5). If $\gamma = 0$, then we also assume that the branch value set B_f has cardinality at least 4; otherwise there is no possible degeneration to be done (if B_f has cardinality 3, then (S, f) is a Belyi pair which is definable over $\overline{\mathbb{Q}}$ as a consequence of Belyi's theorem [10]). Let us consider a collection $\mathcal{F}$ of pairwise disjoint simple loops $\gamma_1, \ldots, \gamma_s \subset R - B_f$ such that the Euler characteristic of each connected component of $R - (B_f \cup \gamma_1 \cup \cdots \cup \gamma_s)$ is negative and none of the components of $R - (\gamma_1 \cup \cdots \cup \gamma_s)$ is a disc with only two branch values, both with branch order equal to 2. Let $\mathcal{G} \subset S$ be the collection of (necessarily simple) loops obtained by lifting those in $\mathcal{F}$ by f. Next, we proceed to identify all points belonging to the same loop in $\mathcal{G}$ to obtain a stable surface S^* of the same genus as S. Similarly, by doing the same procedure to the loops of $\mathcal{F}$, we obtain an stable genus γ orbifold R^*. The map f induces a continuous map $f^* : S^* \to R^*$ of degree n. We call such a pair (S^*, f^*) a *topological stable(γ, n)-gonal pair*. Let us denote by $N(S^*) \subset S^*$ and by $N(R^*) \subset R^*$ the subsets of nodes of S^* and R^*, respectively. Now, if we provide analytically finite Riemann orbifold structures to each of the components of $R^* - N(R^*)$, then we may lift these Riemann orbifold structures under f^* to obtain an analytically finite Riemann orbifold structure on each component of $S^* - N(S^*)$. In this way, $f^* : S^* \to R^*$ is a degree n continuous map so that its restriction to each connected component of $S^* - N(S^*)$ is meromorphic. The resulting (S^*, f^*) (with the provided analytically finite structures on the components) will be called a *stable (γ, n)-gonal pair (topologically) modelled by* (S^*, f^*). Let us observe that the stable Riemann orbifold structures on (S^*, f^*) depends on the spaces of orbifold structures on each of the connected components of R^*; in fact, the product space of the Teichmüller spaces of these components provide a parametrization of the space of stable Riemann orbifold structures of (S^*, f^*). In particular, if each of components of R^* is an spheres with only 3 marked points, then the structure is unique.

In Sect. 6 we provide uniformizing pairs of stable (γ, n)-gonal pairs using noded Fuchsian groups. This uniformization permits to obtain the following fact.

Theorem 2 *Let (S, f) be a (γ, n)-gonal pair of hyperbolic type and let (Γ, G) be an uniformizing pair of it. Then*

(i) The augmented Teichmüller space $NT(\mathbb{D}, G)$ provides a model for the partial closure of $\mathcal{H}_0(S, f)$ obtained by adding the corresponding equivalence classes of stable (γ, n)-gonal pairs.
(ii) The space parametrizing twisted isomorphic classes of stable (γ, n)-gonal pairs can be identified with the quotient $NT(\mathbb{D}, G)/M(\Gamma, G)$, which has the structure of a compact complex orbifold, where $M(\Gamma, G)$ is the same as in Theorem 1.

The above provides a Kleinian groups description being in a parallel point of view as the description provided in [17] for the case $\gamma = 0$.

An interesting situation is provided for genus zero n-gonal pairs (i.e. rational maps), as the above provides a description of degenerations of rational maps in terms of Kleinian groups (this is somehow related to part of the work done in [7]).

Corollary 1 *Let $R \in \mathbb{C}(z)$ be a rational map of degree $d \geq 2$ and let $B = \{p_1, \ldots, p_r\}$ be its locus of branch values. Let n_j be the minimum common multiple of the local degrees of R at the points in $R^{-1}(p_j)$. Assume that $n_1^{-1} + \cdots + n_r^{-1} < r - 2$. Let G be a Fuchsian group acting on $\mathbb{D}$, such that $\mathbb{D}/G$ is the orbifold being the Riemann sphere and whose conical points set is B and the conical order of p_j is n_j. Then the space of isomorphic classes of rational maps topologically equivalent to R is isomorphic to the Teichmüller space $T(\mathbb{D}, G)$, this being a simply-connected complex manifold of dimension $r-3$. Its partial closure obtained by adding isomorphic classes of geometrical degenerations of it is isomorphic to the augmented Teichmüller space $NT(\mathbb{D}, G)$. Similarly, the corresponding space of twisted isomorphic classes is a complex orbifold of dimension $r - 3$ and its closure obtained by adding the twisted classes of its degenerations is isomorphic to a compact complex orbifold of the same dimension.*

This paper is organized as follows. In Sect. 2 we review some definitions and basic facts on Kleinian groups, in particular of noded Fuchsian groups and some of its properties previously obtained in [29,30]. In Sect. 3 we recall the definition and some facts on quasiconformal deformations of Kleinian groups, in particular, the Teichmüller space of a finitely generated Fuchsian group. In Sect. 4 we recall the concept of noded Beltrami coefficients of Kleinian groups that Alexander Vasil'ev and the author have done in [31], and which permits to construct a partial closure of the quasiconformal deformation space of a Kleinian group. We will be mainly interested, for the purpose of this article, on the case of Fuchsian groups, but we provide the general point of view. In this way, we will recall Bers–Abikoff's augmented Teichmüller space [1–3,13] from the notion of noded quasiconformal deformation theory. In the final two sections we use the previous facts to describe, in terms of Fuchsian groups and noded Fuchsian groups, the unifomizations of (γ, n)-gonal and stable (γ, n)-gonal pairs, respectively.

2 Some preliminaries on Kleinian groups

2.1 Analytically finite Riemann orbifolds

An *analytically finite Riemann orbifold* is given by a closed Riemann surface S of genus $g \geq 0$ (the underlying Riemann surface structure of the Riemann orbifold) together with a finite collection of conical points $x_1, \ldots, x_r \in S$ of orders $2 \leq m_1 \leq m_2 \leq \cdots \leq m_r \leq \infty$, respectively. If x_j has cone order ∞, then this means that it represents a puncture of the orbifold. The *signature* of the orbifold is defined by the tuple $(g; m_1, \ldots, m_r)$ and it is called *hyperbolic* if $2g - 2 + \sum_{i=1}^{r}(1 - m_i^{-1}) > 0$. If $r = 3$ and $g = 0$, then the orbifold is called a *triangular orbifold*. If $r = 0$, then the orbifold is a closed Riemann surface. If $r > 0$ and $m_j = \infty$ for all j, then it is an analytically finite punctured Riemann surface (S minus all the points x_j).

2.2 Noded Riemann surfaces/orbifolds

Let us consider a countable collection $\{\mathcal{O}_j\}_{j \in J}$ of Riemann orbifolds, that is, each $\mathcal{O}_j$ consists of a Riemann surface S_j together a discrete collection of cone points $B_j = \{p_{ji}\} \subset S_j$ and integer values $n_{ji} \geq 2$ (the cone orders of the points p_{ji}). Let us consider a discrete collection of points $\mathcal{E} \subset \bigcup_{j \in J}(S_j - B_j)$ and an order two bijective map $T : \mathcal{E} \to \mathcal{E}$. We proceed to identify the points q and $T(q)$, for each $q \in \mathcal{E}$, to obtain a space X; called a *noded Riemann orbifold*. The points obtained by the identification of the points q and $T(q)$, for $q \in \mathcal{E}$, are called (i) *nodes* of X if $T(q) \neq q$ and (ii) *phantom nodes* if $T(q) = q$. We denote the set of nodes and phantom nodes of X by $N(X)$. Observe that the points in $N(X)$ correspond to punctures on each connected component of $X - N(X)$ and that these components are given by $S_j - (\mathcal{E} \cap S_j)$. The set of cone points of X is given by $B(X) := \bigcup_j B_j$. In the case that every $B_j = \emptyset$ (that is, $\mathcal{O}_j$ is just a Riemann surface), we say that X is a *noded Riemann surface*.

An isomorphism between noded Riemann orbifolds is a homeomorphism that sends cone points to cone points (respecting their orders) which, restricted to the complement of the nodes, is analytic. If there is an isomorphism between two noded Riemann orbifolds/surfaces, then we say that they are isomorphic or conformally equivalent. This, in particular, permits to talk on automorphisms of noded Riemann orbifolds/surfaces.

A noded Riemann surface which is homeomorphic to the space obtained by pinching a non-trivial simple loop on a torus is called a *stable Riemann surface of genus one*. A noded Riemann surface which is homeomorphic to the space obtained by the process of pinching a family (necessarily finite) of disjoint simple closed hyperbolic geodesics on a closed Riemann surface of genus $g \geq 2$ is called a *stable Riemann surface of genus* g. Isomorphic classes of stable Riemann surfaces are the extra points Mumford needed to add to the moduli space $\mathcal{M}_g$ of closed Riemann surfaces of genus g to provide the Deligne–Mumford's compactification $\overline{\mathcal{M}_g}$ [22].

Let us observe that the space obtained as the quotient X/H, where X is a noded Riemann surface and H is a (finite) group of automorphisms of X is an example of a noded Riemann orbifold. In the case that X is a stable Riemann surface, then X/H is also called a *stable Riemann orbifold*.

2.3 Kleinian groups

Generalities on Kleinian groups can be found, for instance, in the books [9,41]. Below, we recall some of the needed facts for this paper. A *Kleinian group* is a discrete subgroup G of $\mathrm{PSL}_2(\mathbb{C})$ (seen as the group of Möbius transformations acting on $\widehat{\mathbb{C}}$), its *region of discontinuity* of a Kleinian group G is the open set (which it might be empty) $\Omega(G)$ of points over which G acts properly discontinuous and $\Lambda(G) = \widehat{\mathbb{C}} - \Omega(G)$ is its *limit set*. If $\Omega(G) \neq \emptyset$, then the quotient space $\Omega(G)/G$ is a union of Riemann orbifolds and, by Ahlfor's finiteness theorem [4], if G is also finitely generated, then such a quotient consists of a finite number of analytically finite Riemann orbifolds.

Let us fix some finitely generated Kleinian group G with non-empty region of discontinuity. A loxodromic transformation $\delta \in G$ is called *primitive* if it is not a nontrivial positive power of another loxodromic transformation in G. We say that δ is *simple loxodromic* if there is a simple arc on $\Omega(G)$ which is invariant under δ, called an *axis* of δ, and whose projection on $\Omega(G)/G$ is a simple loop or a simple arc connecting two conical points of order 2. A parabolic transformation $\eta \in G$, with fixed point p, is called *double-cusped*, if (i) any parabolic element of G commuting with η belongs to the cyclic group $\langle \eta \rangle$, and (ii) there are two tangent open discs at p in $\Omega(G)$ whose union is invariant under the stabilizer of p in G. The *extended region of discontinuity* of G is defined as $\Omega(G)^{ext} = \Omega(G) \cup P(G)$, where $P(G)$ is the set of fixed points of the double-parabolic elements of G. On $\Omega(G)^{ext}$ we consider its cuspidal topology; the topology generated by the usual open sets in $\Omega(G)$ and the sets of the form $D_1 \cup D_2 \cup \{p\}$, where $p \in P(G)$, and $D_1, D_2 \subset \Omega(G)$ are round discs tangent at p. Observe that if G has no parabolic transformations, then $\Omega(G) = \Omega(G)^{ext}$. In the cuspidal topology, the group G acts as a group of homeomorphisms on $\Omega(G)^{ext}$, keeping invariant each $\Omega(G)$ and $P(G)$, and its restriction to $\Omega(G)$ being by holomorphic automorphisms.

Lemma 1 *Let G be a finitely generated Kleinian group with $\Omega(G) \neq \emptyset$. Then $S = \Omega^{ext}(G)/G$ is a noded Riemann surface, its set of nodes is $P(G)/G$ (the phantom nodes given by those parabolic fixed points being fixed by an involution in G).*

Proof By Selberg's lemma, G contains a torsion-free finite index normal subgroup K. The finite index condition asserts that $\Omega(K) = \Omega(G)$ and $P(K) = P(G)$, in particular, $\Omega^{ext}(K) = \Omega^{ext}(G)$. The quotient $\Omega(G)^{ext}/K$ is a noded Riemann surface whose nodes corresponds one-to-one to the K-equivalence classes of the parabolic fixed points in $P(G)$ (no phantom nodes in this case). By Ahlfors' finiteness theorem, the number of components of $\Omega(G)/K = \Omega(G)^{ext}/K - N(\Omega(G)^{ext}/K)$ is finite, each one an analytically finite Riemann surface. It follows that $\Omega(G)^{ext}/G$, with the quotient cuspidal topology, is a noded Riemann orbifold, it contains the orbifold $\Omega(G)/G$ as a dense open subset, and its set of nodes is $P(G)/G$ (those points in $P(G)$ having an involution in their stabilizers produce the phantom nodes). □

2.4 Fuchsian groups

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathrm{Aut}(\mathbb{D}) \cong \mathrm{PSL}_2(\mathbb{R})$ be its group of conformal automorphisms. A *Fuchsian group* is a finitely generated Kleinian group being a subgroup of $\mathrm{Aut}(\mathbb{D})$. The Fuchsian group is called of the *first kind* if its limit set is all of the unit circle (so is region of discontinuity consists of two discs); otherwise, it is called of the *second kind* (so its region of discontinuity is connected).

As a consequence of Selberg's lemma [9], a Fuchsian group F has a torsion-free normal subgroup K of finite index; this is again a finitely generated Fuchsian group of the same kind as F. If F is of the second kind without parabolic elements, then K is a Schottky group of some finite rank $g \geq 0$ and if F is of the first kind without parabolic elements, then K is a co-compact Fuchsian group uniformizing a closed Riemann surface of some genus $g \geq 2$ (in this case we say that K is a π^g group).

By classical work of Fricke and Klein, a Fuchsian group F of the first kind and without parabolic elements has a presentation of the form

$$F = \left\langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \delta_1, \ldots, \delta_r : \prod_{j=1}^{g} [\alpha_j, \beta_j] \prod_{i=1}^{r} \delta_i = \delta_1^{m_1} = \cdots = \delta_r^{m_r} = 1 \right\rangle,$$

where $[\alpha_j, \beta_j] = \alpha_j \beta_j \alpha_j^{-1} \beta_j^{-1}$, $m_i \geq 2$ are integers so that $2g - 2 + \sum_{i=1}^{r}(1 - m_i^{-1}) > 0$. In this case, $\mathbb{D}/F$ is a compact hyperbolic Riemann orbifold of signature $(g; m_1, \ldots, m_r)$; which is also called the signature of F. If $r = 0$, then $\mathbb{D}/F$ is a closed Riemann surface of genus g and its signature is denoted by $(g; -)$. As a consequence of the uniformization theorem, every compact hyperbolic orbifold $\mathcal{O}$ is isomorphic to $\mathbb{D}/F$ for some Fuchsian group F.

2.5 Noded Fuchsian groups

A geometrically finite Kleinian group which isomorphic to some Fuchsian group of the first kind is called a *noded Fuchsian group*. In particular, it cannot have rank two parabolic subgroups and all of its parabolic elements are double-cusped.

Theorem 3 [30] *Noded Fuchsian groups have non-empty region of discontinuity.*

Torsion-free noded Fuchsian groups come in two flavors [29,30]: (i) *noded Schottky groups of rank* $g \geq 0$ (isomorphic to free groups of rank $g \geq 0$) and (ii) *noded* π^g *groups* (isomorphic to the fundamental group of a closed orientable surface of genus $g \geq 2$). As a consequence of Selberg's lemma, every noded Fuchsian group G has a finite index normal subgroup K being either a noded Schottky group or a noded π^g

group. The finite index condition asserts that $\Omega(K) = \Omega(G)$ and $P(K) = P(G)$, in particular, $\Omega^{ext}(K) = \Omega^{ext}(G)$.

2.6 Noded retrosection theorem

Koebe's retrosection theorem [36] asserts that every closed Riemann surface of genus g can be uniformized by a Schottky group of rank g. A retrosection theorem with nodes hold and it can be stated as follows.

Theorem 4 (Noded retrosection theorem [29,30])

1. *If G is a noded Schottky group of rank g, then $\Omega(G)^{ext}$ is connected and $\Omega(G)^{ext}/G$ is a stable Riemann surface of genus g.*
2. *If S is a stable Riemann surface of genus g, then there is a noded Schottky group G of genus g such that $\Omega(G)^{ext}/G$ is conformally equivalent to S.*

Remark 1 The region of discontinuity of a Schottky group is always connected, but that of a noded Schottky group is not in general; connectivity only holds for its extended region of discontinuity.

Corollary 2 *If G is a noded Fuchsian group containing a noded Schottky group as a finite index normal subgroup, then $\Omega(G)^{ext}$ is connected and $\Omega(G)^{ext}/G$ consists of an stable Riemann orbifold.*

2.7 Simultaneous uniformization theorem

If G is a torsion free purely loxodromic quasi-Fuchsian group (i.e., a quasiconformal deformation of a torsion free Fuchsian group of the first kind without parabolics), then its region of discontinuity consists of two topological discs, say D_1 and D_2, and the quotients D_1/G and D_2/G are closed Riemann surfaces of the same genus $g \geq 2$. Bers' simultaneous uniformization theorem [11] asserts that, given any two closed Riemann surfaces S_1 and S_2 of the same genus $g \geq 2$, then it is possible to find G as above so that we may assume that S_j is isomorphic to D_j/G, for $j = 1, 2$.

Let G be now a noded π^g group ($g \geq 2$). If G is purely loxodromic, then Maskit [43] has shown that G is in fact a quasi-Fuchsian group and we are as above. So, we are left to consider the case when G contains parabolic transformations. In [30,43] there provided examples of noded π^g groups (with parabolic transformations) which are not quasi-Fuchsian ones (the example in [43] is a B-group [41] and that in [30] is a group without invariant components in its region of discontinuity). But, if consider the extended region of discontinuity, then the situation is similar to the quasi-Fuchsian case, as seen in the following.

Theorem 5 (Simultaneous uniformization theorem with nodes [30,39])

1. *If $g \geq 2$ and G is a noded π^g group, then $\Omega(G)^{ext}$ consists of exactly two simply-connected invariant components, and $\Omega(G)^{ext}/G$ consists of exactly two stable Riemann surface of genus g.*

2. *If S and R are two stable Riemann surfaces of genus $g \geq 2$, then there is a noded π^g group G such that $\Omega(G)^{ext}/G$ is isomorphic to $S \cup R$.*

Corollary 3 *Let G be a noded Fuchsian group, containing a noded π^g group as a finite index normal subgroup. Then $\Omega(G)^{ext}$ consists of two simply-connected invariant components, and $\Omega(G)^{ext}/G$ consists of exactly two stable Riemann orbifolds.*

The above result was proved by Abikoff [1] for the case of cusps, that is, for regular B-groups.

Remark 2 Let G be a noded Fuchsian group. As seen above (Theorems 4, 5), the noded Riemann surface $\Omega(G)^{ext}/K$ consists of either: (i) two stable Riemann surfaces of genus g, if K is noded π^g-group, or (ii) one stable Riemann surface of genus g, if K is a noded Schottky group of rank g. For instance, the cyclic group $G = \langle \gamma(z) = z + 1 \rangle$ is a noded Schottky group of rank one, $\Omega(G)^{ext} = \widehat{\mathbb{C}}$, $\Omega(G) = \mathbb{C}$ and $\Omega(G)^{ext}/G$ is a stable Riemann surface of genus one (its node being the projection of ∞).

3 Some preliminaries on Teichmüller spaces of Kleinian groups

In this section, G will be a finitely generated Kleinian group, with non-empty region of discontinuity $\Omega(G)$, and $\Delta \subset \Omega(G)$ will be a fixed non-empty G-invariant collection of components of $\Omega(G)$ (that is, every $\gamma \in G$ permutes its connected components). We proceed to recall the definition of the quasiconformal deformation space $T(\Delta, G)$ of G supported at Δ. If G is a Fuchsian group of the first kind acting on the unite disc $\mathbb{D}$, then $T(\mathbb{D}, G)$ provides of a model for the Teichmüller space of the Riemann orbifold $\mathbb{D}/G$.

3.1 Quasiconformal homeomorphisms

Let $L^\infty(\widehat{\mathbb{C}})$ be the Banach space of measurable maps $\mu : \widehat{\mathbb{C}} \to \mathbb{C}$, with the essential supreme norm $\|\cdot\|_\infty$, and let $L_1^\infty(\widehat{\mathbb{C}})$ be its unit ball. An orientation preserving homeomorphism $w : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is called a *quasiconformal homeomorphism* if there is some $\mu \in L_1^\infty(\widehat{\mathbb{C}})$ (called a *complex dilatation* of w) such that w has distributional partial derivatives ∂w, $\overline{\partial} w$ in $L^2_{loc}(\widehat{\mathbb{C}})$ satisfying the Beltrami equation

$$\overline{\partial} w(z) = \mu(z) \partial w(z), \quad \text{a.e. } z \in \widehat{\mathbb{C}},$$

where $L^2_{loc}(\widehat{\mathbb{C}})$ means L^2 on compacts sets of $\widehat{\mathbb{C}}$. The existence and uniqueness of quasiconformal homeomorphisms is due to Morrey [48].

Theorem 6 (Measurable Riemann mapping theorem [6,48]) *If $\mu \in L_1^\infty(\widehat{\mathbb{C}})$, then there exists, and it is unique, a quasiconformal homeomorphism $w_\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, with complex dilation μ, and fixing ∞, 0 and 1. Moreover, if μ vary continuously or real-analytically or holomorphically and z_0 is a fixed point, then $w_\mu(z_0)$ varies also in the same way.*

3.2 Beltrami coefficients for *G* supported at Δ

The discreteness property of G asserts that

$$L^{\infty}(\Delta, G) = v \Big\{ \mu \in L^{\infty}(\widehat{\mathbb{C}}) : \mu(\gamma(z))\overline{\gamma'(z)} = \mu(z)\gamma'(z), \text{ a.e. } \Delta, \\ \forall \gamma \in G,\ \mu(z) = 0,\ z \in \Delta^{c} \Big\}$$

is a closed subspace of $L^{\infty}(\widehat{\mathbb{C}})$; so it is a Banach space with essential supreme norm $\|\cdot\|_{\infty}$. Its unit ball is

$$L_{1}^{\infty}(\Delta, G) = L^{\infty}(\Delta, G) \cap L_{1}^{\infty}(\widehat{\mathbb{C}}) = \{\mu \in L^{\infty}(\Delta, G);\ \|\mu\|_{\infty} < 1\}.$$

The measurable functions in $L_{1}^{\infty}(\Delta, G)$ are called the *Beltrami coefficients for Gsupported inΔ*. The following lemma is a classical result and its proof can be found, for instance, in [40].

Lemma 2 *Let $\mu \in L_{1}^{\infty}(\Delta, G)$ and let $w : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasiconformal homeomorphism with complex dilatation μ. Then wGw^{-1} is a Kleinian group with the region of discontinuity $w(\Omega(G))$. Moreover, $w(\Delta)$ is a WGw^{-1}-invariant collection of components.*

3.3 Teichmüller and moduli spaces

By the measurable Riemann mapping theorem, if $\mu \in L_{1}^{\infty}(\Delta, G)$ and w_1, w_2 are quasiconformal homeomorphisms, both with complex dilation μ, then there is a Möbius transformation $A \in \mathrm{PSL}_2(\mathbb{C})$ so that $w_2 = Aw_1$.

Let $\mu, \nu \in L_{1}^{\infty}(\Delta, G)$ be two Beltrami coefficients for G supported in Δ. If w_{μ} (respectively w_{ν}) is a quasiconformal homeomorphism with complex dilation μ (respectively ν), then (by Lemma 2) there is a natural isomorphism $\theta_{\mu} : G \to w_{\mu}Gw_{\mu}^{-1}$ (respectively $\theta_{\nu} : G \to w_{\nu}Gw_{\nu}^{-1}$). We say that μ and ν are *Teichmüller equivalent* (respectively, *isomorphic*) if there is $A \in \mathrm{PSL}_2(\mathbb{C})$ with $\theta_{\mu}(\gamma) = A\theta_{\nu}(\gamma)A^{-1}$, for all $\gamma \in G$ (respectively, $\theta_{\mu}(G) = A\theta_{\nu}(G)A^{-1}$). The space $T(\Delta, G)$ (respectively, $\mathcal{M}(\Delta, G)$) of Teichmüller (respectively, isomorphic) equivalence classes of Beltrami coefficients for G supported in Δ is called the *Teichmüller space* (respectively, *moduli space*) *of G supported in Δ*. The *modular group of Gsupported atΔ* is the group $M(\Delta, G)$ given by the isotopy classes of quasiconformal homeomorphisms $w : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, with complex dilation in $L_{1}^{\infty}(\Delta, G)$, so that $wGw^{-1} = G$. There is the natural action

$$M(\Delta, G) \times T(\Delta, G) \to T(\Delta, G) : ([w], [\mu]) \mapsto [\nu],$$

where $\nu \in L_{1}^{\infty}(\Delta, G)$ is complex dilation of the quasiconformal homeomorphism $w_{\mu}w^{-1}$.

It is well known that $T(\Delta, G)$ is a finite dimensional complex manifold (simply-connected if Δ is connected and simply-connected), that $M(\Delta, G)$ acts as a discrete group of its holomorphic automorphisms and that $\mathcal{M}(\Delta, G) = T(\Delta, G)/M(\Delta, G)$ is a complex orbifold of same dimension as $T(\Delta, G)$ (generalities on this can be found, for instance, in [12,37,45,49]).

3.4 The Fuchsian case

Let us assume that G is a Fuchsian group of the first kind acting on the unit disc $\mathbb{D}$; so the Riemann orbifold $\mathbb{D}/G$ has signature $(\gamma; m_1, \ldots, m_r)$. In this case $T(\mathbb{D}, G)$ is a simply-connected complex manifold of dimension $3\gamma - 3 + r$ (see, for instance, the book [49]) and, moreover, this space is a model for the Teichmüller space of the Riemann orbifold $\mathbb{D}/G$. If Γ is a finite index subgroup of G, then the natural inclusion $L_1^\infty(\mathbb{D}, G) \subset L_1^\infty(\mathbb{D}, \Gamma)$ induces a holomorphic embedding $T(\mathbb{D}, G) \subset T(\mathbb{D}, \Gamma)$. Also, in this case, the subgroup $M(\Gamma, G)$ of $M(\mathbb{D}, G)$, formed by those isotopy classes of quasiconformal homeomorphisms for which $w\Gamma w^{-1} = \Gamma$, has finite index. So the complex orbifold $T(\mathbb{D}, G)/M(\Gamma, G)$ provides a finite degree branched cover of moduli space $\mathcal{M}(\mathbb{D}, G)$.

Remark 3 Let G a finitely generated Kleinian group and let Δ a G-invariant collection of connected components of $\Omega(G) \neq \emptyset$. Let $\Delta_1, \ldots, \Delta_r$ be a maximal collection of non-G-conjugate components of Δ. Let G_j be the G-stabilizer of Δ_j and let $G(\Delta_j)$ be the union of all components of Δ which are G-conjugate to Δ_j. As a consequence of results of Kra [37] and Sullivan [55], it was observed by Kra and Maskit [38] that

$$T(\Delta, G) = T(G, G(\Delta_1)) \times \cdots \times T(G, G(\Delta_r))$$

and that its universal cover space is $T(\mathbb{D}, F_1) \times \cdots \times T(\mathbb{D}, F_r)$, where F_j is a Fuchsian group acting on $\mathbb{D}$ so that $\mathbb{D}/F_j \cong \Delta_j/G_j$. This in particular asserts that that the previous statement done for the case of Fuchsian groups still valid for G.

4 Noded quasiconformal deformation spaces of Kleinian groups

If G is a Fuchsian group of the first kind acting on the unit disc $\mathbb{D}$, many different compactifications of $T(\mathbb{D}, G)$ have been provided. For instance, Bers' compactification [13] is obtained by holomorphically embedding $T(\mathbb{D}, G)$ into the space of quadratic holomorphic forms on $\mathbb{D}/G$ (this being a finite dimensional complex vector space) and taking its closure in there, and Thurston's compactification [56] is obtained by taking hyperbolic lengths at simple closed geodesics, which provides a holomorphic embedding into an infinite-dimensional projective space. On Ber's compactifications it is no possible to extend continuously the action of the corresponding modular group. There is not a natural relation between these compactifications and Deligne–Mumford's compactification. A partial closure of $T(\mathbb{D}, G)$, called the *augmented Teichmüller space* $\widehat{T}(\mathbb{D}, G)$, was constructed by Bers [13] (see also Abikoff's paper [3]). This space is a non-compact Hausdorff space over which the modular group $M(\mathbb{D}, G)$ extends continuously and such that $\widehat{T}(\mathbb{D}, G)/M(\mathbb{D}, G)$ coincides with Deligne–Mumford's compactification [33]. Other classical references to quotients of the augmented Teichmüller space and deformation spaces of Riemann surfaces can be found, for instance, in [14,24]. The added points to $\widehat{T}(\mathbb{D}, G)$ are certain regular B-groups (geometrically finite Kleinian groups with a simply-connected invariant connected component of its region of discontinuity [41]).

If Δ is a collection of G-invariant components of $\Omega(G)$, where G is a finitely generated Kleinian group, then in [31] we have constructed a partial closure $NT(\Delta, G)$ of $T(\Delta, G)$ such that the boundary points correspond in a natural way to the boundary points in the Deligne–Mumford compactification of moduli space of Δ/G. If G is a Fuchsian group of the first kind and $\Delta = \mathbb{D}$, we have $NT(\mathbb{D}, G) = \widehat{T}(\mathbb{D}, G)$. The extra points we add to obtain $NT(\Delta, G)$ produce, in terms of Kleinian groups, stable Riemann orbifolds and they correspond to deformations of the group G by the process of approximation of double-cusped parabolic elements of G by certain *primitive simple loxodromic* ones (see the works of Keen et al. [35], and of Maskit in [42,44]). The deformation is produced by some boundary points of $L_1^{\infty}(G, \Delta)$, called *noded Beltrami differentials for* G. At the level of the Riemann orbifold Δ/G this means that we permit certain pairwise disjoint simple closed geodesics (and maybe some simple geodesic arcs connecting conical values of order 2) to degenerate to points in order to produce a finite collection of noded Riemann orbifolds. The loops and arcs which we consider in the degeneration process are the projections of appropriately chosen axes of the primitive simple loxodromic elements of the group that approach doubly-cusped parabolic transformations.

Next, we will assume that G is a finitely generated Kleinian group, with non-empty region of discontinuity, and that Δ a non-empty G-invariant collection of connected components of its region of discontinuity, and we proceed to recall the construction of $NT(\Delta, G)$ as done in [31].

4.1 Region of discontinuity of Beltrami forms

For each $\mu \in L^{\infty}(\widehat{\mathbb{C}})$ we define its *region of discontinuity* $\Omega(\mu)$ as the set of all points $p \in \widehat{\mathbb{C}}$ for which there is an open set U, containing p, such that $\|\mu|_U\|_{\infty} < 1$. The set $\Lambda(\mu) = \widehat{\mathbb{C}} - \Omega(\mu)$ is the *limit set* of μ. By the definition, $\Omega(\mu)$ is open (it might be empty) and $\Lambda(\mu)$ is compact.

If $\mu \in \overline{L_1^{\infty}(\Delta, G)}$, the closure of $L_1^{\infty}(\Delta, G)$ in $L^{\infty}(\Delta, G)$, then in [31] it was observed that both $\Omega(\mu)$ and $\Lambda(\mu)$ are $G-$invariant. Also, by the G-invariance of $\Omega(\mu)$ and $\Omega(G)$, the open set $\Omega := \Omega(\mu) \cap \Omega(G)$ is G-invariant. Observe that, as μ is equal to zero on the complement of Δ, all connected components of $\Omega(G) - \Delta$ are necessarily contained in Ω.

4.2 Noded quasiconformal maps for (Δ, G)

If $V \subset \widehat{\mathbb{C}}$ is a non-empty open set, then $L_{loc}^{2,1}(V)$ denotes the complex vector space of maps $w : V \to \widehat{\mathbb{C}}$ with locally integrable distributional derivatives. Let $\mu \in \overline{L_1^{\infty}(\Delta, G)}$, where $\Omega(\mu) \neq \emptyset$. An orientation-preserving map $w \in L_{loc}^{2,1}(\Omega(\mu))$ is a *noded quasiconformal map for* (Δ, G) with dilatation μ if the following hold:

(i) there is a component of $\Omega(\mu)$ homeomorphically mapped by w onto its image;
(ii) $\overline{\partial} w(z) = \mu(z)\partial w(z)$, a.e. in $\Omega(\mu)$;
(iii) there is a sequence $\mu_n \in L_1^\infty(\Delta, G)$, converging to μ almost everywhere in $\Omega(\mu)$;
(iv) there is a sequence $w_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of quasiconformal homeomorphisms with complex dilatations μ_n, converging to w locally uniformly in $\Omega(\mu)$.

In the above definition there are many properties to be checked, but the following existence result is classical and a proof can be find in [31].

Proposition 1 *Let $\mu \in \overline{L_1^\infty(\Delta, G)}$ be such that $\Omega(\mu) \neq \emptyset$, Ω_1 be a connected component of $\Omega(\mu)$ and $x_1, x_2, x_3 \in \Omega_1$ three different points. Then there is a noded quasiconformal map $w : \Omega(\mu) \to \widehat{\mathbb{C}}$, for (Δ, G), with dilatation μ, fixing the points x_1, x_2 and x_3, which is a homeomorphism when restricted to Ω_1.*

Remark 4 Let us observe that the noded quasiconformal map w for (Δ, G) in Proposition 1 might be constant on some other connected component of $\Omega(\mu)$ different from Ω_1.

4.3 Noded family of arcs

A countable collection $\mathcal{F}_\mu = \{\alpha_1, \alpha_2, \ldots\}$ of pairwise disjoint simple arcs (including end points) is called a *noded family of arcs associated with* $\mu \in \overline{L_1^\infty(G, \Delta)}$, if the following properties hold:

(1) for each n, $\alpha_n^* \subset \Delta$, where α_n^* denotes α_n minus both extremes (the extremes belong to the boundary of Δ);
(2) the spherical diameter of α_n goes to 0 as n goes to ∞;
(3) $\Lambda(\mu) = \overline{\cup_{j=1}^\infty \alpha_j}$;
(4) $\Omega(\mu) \subset \widehat{\mathbb{C}}$ is a dense subset;
(5) for each n, the group $G_n = \{g \in G; g(\alpha_n) = \alpha_n\}$, is either a cyclic loxodromic group or a $\mathbb{Z}_2$-extension of a cyclic loxodromic group.

Remark 5 There exist examples of $\mu \in \overline{L_1^\infty(G, \Delta)}$ admitting no associated noded family of arcs. In [31] there are constructed examples in the positive direction.

4.4 Noded Beltrami coefficients for *G* supported in Δ

By the stereographic projection, we may see the Riemann sphere as the unit sphere in $\mathbb{R}^3$. This allows us to consider the spherical metric and work with the spherical diameter of a subset of $\widehat{\mathbb{C}}$. An element $\mu \in \overline{L_1^\infty(\Delta, G)}$ will be called a *noded Beltrami coefficient for G supported in*Δ *if:*

(I) μ has a noded family of arcs $\mathcal{F}_\mu = \{\alpha_1, \alpha_2, \ldots\}$, and
(II) there is a continuous map $w : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, called a *noded quasiconformal deformation of G with complex dilatation* μ, satisfying the following properties:

   (1) w is injective in $\widehat{\mathbb{C}} - \mathcal{F}_\mu$;

(2) $w : \Omega(\mu) \to \widehat{\mathbb{C}}$ is a noded quasiconformal map for (Δ, G) with complex dilatation μ;
(3) the restriction of w to each arc α_i is a constant p_i, where $p_i \neq p_j$ for $i \neq j$.

We denote by $L^{\infty}_{\text{noded}}(\Delta, G)$ the subset of $\overline{L^{\infty}_1(\Delta, G)}$ consisting of the noded Beltrami coefficients for G supported at Δ. It is not difficult to see that $L^{\infty}_1(\Delta, G) \subset L^{\infty}_{\text{noded}}(\Delta, G)$.

Remark 6 Let $\mu \in L^{\infty}_{\text{noded}}(\Delta, G)$, with noded family of arcs $\mathcal{F}$, and let $w : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a noded quasiconformal deformation of G with the complex dilatation μ. Then, the following statements are true

(1) if $\alpha \in \mathcal{F}_{\mu}$, then both end points are the fixed points of a loxodromic element of G. Such a loxodromic element keeps invariant the connected component of $\Omega(G)$ containing α;
(2) if $\Lambda(\mu) \neq \emptyset$, then $\Lambda(G) \subset \Lambda(\mu)$. This is a consequence of Proposition E.4 in [41, page 96];
(3) $w(\Omega(u)) \cap w(\Lambda(\mu)) = \emptyset$. Indeed, if there were points $p_1 \in \Omega(\mu)$ and $p_2 \in \Lambda(\mu)$, such that $w(p_1) = w(p_2) = q$, then by continuity of w we could find two disjoint open sets $U \subset \Omega(\mu)$ and V, $p_1 \in U$, and $p_2 \in V$, such that $w(U) = w(V)$. The density property of $\Omega(\mu)$ asserts that there are points $q_1 \in U$ and $q_2 \in V \cap \Omega(\mu)$ for which $w(q_1) = w(q_2)$, contradicting the injectivity of the map w restricted to $\Omega(\mu)$;
(4) As a consequence of the definition of noded Beltrami differential, we have $w(\widehat{\mathbb{C}}) = \widehat{\mathbb{C}}$, that is, the map w is surjective.

The importance of the noded quasiconformal deformations of Kleinian groups is reflected in the following result.

Theorem 7 [31] *Let $\mu \in L^{\infty}_{\text{noded}}(\Delta, G)$ and $w : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a noded quasiconformal deformation for G with the complex dilatation μ. Then there is a unique Kleinian group $\theta(G)$ and a unique isomorphism of groups $\theta : G \to \theta(G)$ such that $w \circ \gamma = \theta(\gamma) \circ w$. Moreover, the region of discontinuity of $\theta(G)$ is $w(\Omega(\mu) \cap \Omega(G))$.*

The proof of the above theorem provided in [31] also permit to have the following more general situation.

Corollary 4 [31] *Let $\mu \in \overline{L^{\infty}_1(\Delta, G)}$ and let $w : \Omega(\mu) \to w(\Omega(\mu))$ be a noded quasiconformal homeomorphism with complex dilatation μ. Suppose that there is a sequence w_n of quasiconformal homeomorphisms with corresponding complex dilatations $\mu_n \in L^{\infty}_1(\Delta, G)$, converging locally uniformly to w in $\Omega(\mu)$. Then there exist a group $\theta(G)$ of Möbius transformations and an isomorphism of groups $\theta : G \to \theta(G)$, such that $w \circ \gamma = \theta(\gamma) \circ w$.*

The following result relates two noded quasiconformal deformations with the same dilation similar as the situation for the quasiconformal ones.

Proposition 2 *Let w_1 and w_2 be noded quasiconformal deformations of G with the same complex dilatation $\mu \in L^{\infty}_{noded}(\Delta, G)$. Then there exists an orientation preserving homeomorphism $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, whose restriction $T : w_1(\Omega(\mu)) \to w_2(\Omega(\mu))$ is a conformal mapping, such that $T \circ w_1 = w_2$.*

Proof The construction of T is given as follows. (3.1) If $x \in w_1(\Omega(\mu))$, then set $T(x) = w_2(w_1^{-1}(x))$. (3.2) If $x \in w_1(\alpha_n)$, then set $T(x) = w_2(\alpha_n)$. (3.3) If $x \in \Lambda(\mu) - \cup_n \alpha_n([0, 1])$, then set $T(x) = w_2(w_1^{-1}(x))$. □

Remark 7 Let $\mu \in L^{\infty}_{\text{noded}}(\Delta, G)$, with associated noded family of arcs $\mathcal{F}$, and let $w : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a noded quasiconformal deformation of G with complex dilatation μ. If $\theta(G)$ and $\theta : G \to \theta(G)$ are as in Theorem 7, then the following hold.

(1) If p belongs to one of the arcs in $\mathcal{F}$, then $w(p)$ is necessarily a doubly-cusped parabolic fixed point of $\theta(G)$.
(2) If $p \in \Lambda(G)$ is a loxodromic fixed point in G, which does not belong to any arc of $\mathcal{F}$, then $w(p)$ is again a loxodromic fixed point of $\theta(G)$.
(3) If p is a rank two parabolic fixed point of G, then $w(p)$ is again a rank two parabolic fixed point in $\theta(G)$.
(4) If p is a doubly-cusped parabolic fixed point of G, then $w(p)$ is again doubly-cusped in $\theta(G)$.

The previous remark permits to see the following fact.

Theorem 8 *Under the hypothesis of Theorem 7, if G is geometrically finite, then $\theta(G)$ is also geometrically finite.*

The above, applied to Fuchsian groups of the first kind, permits to obtain the following with respect to noded Fuchsian groups.

Corollary 5 *Let G be a Fuchsian group of the first kind. Let $\mu \in L^{\infty}_{noded}(\mathbb{D}, G)$, let $w : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a noded quasiconformal deformation for G with the complex dilation μ and let $\theta : G \to \theta(G)$ such that $w \circ \gamma = \theta(\gamma) \circ w$, for $\gamma \in G$. Then $\theta(G)$ is a noded Fuchsian group.*

4.5 Topological realizations

Let G be a finitely generated Kleinian group and Δ a non-empty collection of G-invariant components of its (non-empty) region of discontinuity.

A simple loop $\alpha \subset S = \Delta/G$ (or a simple arc connecting two branch values of order 2) is called *pinchable* if (a) its lifting on $\Omega(G)$ consists of pairwise disjoint simple arcs, and (b) each of these components is stabilized by a primitive loxodromic transformation in G. The stabilizer of a component of the lifting of a pinchable loop α is a cyclic group generated by a primitive loxodromic transformation, and the stabilizer of a component of the lifting of a pinchable arc is a $\mathbb{Z}/2\mathbb{Z}$-extension of a cyclic group generated by a primitive loxodromic transformation. We say that such a cyclic group (or $\mathbb{Z}/2\mathbb{Z}$-extension) is defined by the pinchable loop α. If β_1 and β_2 are two components of the lifting of a pinchable α, then their stabilizers are conjugate in G. We say that a collection $\{\alpha_1, \ldots, \alpha_m\}$ of pairwise disjoint pinchable loops or arcs is admissible if they define non-conjugate groups in G for $i \neq j$.

Remark 8 In the particular case that G is a Fuchsian group of the first kind and $\Delta = \mathbb{D}$, then every homotopically non-trivial loop in Δ/G avoiding the branch locus is pinchable. Moreover, every collection $\mathcal{F}$ of pairwise disjoint pinchable loops is admissible.

Also, if Γ is a finite index subgroup of G and $Q : \mathbb{D}/\Gamma \to \mathbb{D}/G$ is a branched cover induced by the inclusion $\Gamma < G$, then the collection $Q^{-1}(\mathcal{F})$ is admissible for Γ.

For each admissible collection $\mathcal{F} = \{\alpha_1, \ldots, \alpha_m\}$ of pinchable loops on S we define the following equivalence relation on the Riemann sphere $\widehat{\mathbb{C}}$. Two points $p, q \in \widehat{\mathbb{C}}$ are equivalent if either:

(1) $p = q$; or
(2) there is a component $\widetilde{\alpha_j}$ of the lifting of some pinchable loop or arc α_j, such that $p, q \in \widetilde{\alpha_j} \cup \{a, b\}$, where a and b are the endpoints of $\widetilde{\alpha_j}$ (that is, the fixed points of a primitive loxodromic transformation in the stabilizer of $\widetilde{\alpha_j}$ in G)

The set of equivalence classes for such an equivalence relation is topologically the Riemann sphere. In fact, let us denote by $\widetilde{\mathcal{F}}$ the collection of all arcs (including their endpoints), as considered in (2) above, and let us consider the collection of continua given by the collection of arcs in $\widetilde{\mathcal{F}}$ as points and also each of the points in the complement of $\widetilde{\mathcal{F}}$. The discreteness of G asserts that such collections of points is a semi-continuous collection of points, as defined in [47], and the result now follows from [47, Thm2]. Let us denote by $P : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ the natural continuous projection defined by the above relation. As a consequence of the results in [44] we have the following fact.

Proposition 3 *There exists an orientation preserving homeomorphism $Q : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, such that the map $Q \circ P : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a noded quasiconformal deformation of the group G.*

As a consequence of the above, the geometrically finite Kleinian groups constructed in [35] are obtained by noded quasiconformal deformations of the suitable Kleinian groups. The following topological realization was seen in [31].

Proposition 4 [31] *Let G be a Kleinian group and $\mathcal{F}$ be an admissible collection of pinchable loops. Denote by $P : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ the continuous projection naturally induced by the equivalence relation defined by $\mathcal{F}$. Then, there is a discrete group $\theta(G)$ of orientation preserving homeomorphisms of the Riemann sphere and there is an isomorphism $\theta : G \to \theta(G)$, such that $P\gamma = \theta(\gamma)P$, for all $\gamma \in G$.*

Remark 9 Results of [44,52] assert that, in Proposition 4, if G is geometrically finite, then $\theta(G)$ is also geometrically finite. In fact, it can be shown that θ and $\theta(G)$ are as obtained in Theorem 7, so the geometrically finiteness also follows from Theorem 8.

4.6 The noded Teichmüller space of *G* supported in Δ

We may extend the Teichmüller equivalence relation, given previously on $L_1^{\infty}(\Delta, G)$, to the whole $L_{\text{noded}}^{\infty}(\Delta, G)$ as follows. Let $\mu, \nu \in L_{\text{noded}}^{\infty}(\Delta, G)$ and w_μ, w_ν be associated noded quasiconformal deformations for G, respectively. Theorem 7 asserts the existence of isomorphisms

$$\theta_\mu : G \to G_\mu \quad \text{and} \quad \theta_\nu : G \to G_\nu,$$

where G_μ and G_ν are Kleinian groups such that $w_\mu g = \theta_\mu(g) w_\mu$ and $w_\nu g = \theta_\nu(g) w_\nu$ for all $g \in G$. We say that μ and ν are *noded Teichmüller equivalent* if there is an orientation preserving homeomorphism $A : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, such that:

(1) $A(w_\mu(\Omega(\mu))) = w_\nu(\Omega(\nu))$;
(2) $A : w_\mu(\Omega(\mu)) \to w_\nu(\Omega(\nu))$ is conformal;
(3) $\theta_\nu(g) = A\theta_\nu(g)A^{-1}$ for all $g \in G$.

If in the above we replace (3) by $\theta_\nu(G) = A\theta_\nu(G)A^{-1}$, then we will say that μ and ν are *isomorphic*. The *noded Teichmüller space of G supported in Δ* is the set $NT(\Delta, G)$ of the noded Teichmüller equivalence classes of noded Beltrami coefficients for G supported in Δ. If $\Delta = \Omega(G)$, then we denote it by $NT(G)$.

Remark 10 If $\mu, \nu \in L^\infty_{\text{noded}}(\Delta, G)$ are noded Teichmüller equivalent and $\mu \in L^\infty_1(\Delta, G)$, then (1) and (2) in above definition assert that $\nu \in L^\infty_1(\Delta, G)$ and that they are Teichmüller equivalent. Moreover, the inclusion $L^\infty_1(\Delta, G) \subset L^\infty_{\text{noded}}(\Delta, G)$ induces, under the above equivalence relation, the inclusion $T(\Delta, G) \subset NT(\Delta, G)$.

The modular group $M(\Delta, G)$ extends naturally to act on $NT(\Delta, G)$

$$M(\Delta, G) \times NT(\Delta, G) \to NT(\Delta, G) : ([w], [\mu]) \mapsto [\nu],$$

where $\nu \in L^\infty_{noded}(\Delta, G)$ is complex dilation of the noded quasiconformal deformation $w_\mu w^{-1}$. The quotient space $N\mathcal{M}(\Delta, G) = NT(\Delta, G)/M(\Delta, G)$ is the space of isomorphic classes of noded Beltrami coefficients for G supported at Δ. It provides Deligne–Mumford's compactification of the moduli space $\mathcal{M}(\Delta, G)$.

Remark 11 In the setting of representation of groups, $NT(\Omega(G), G)$ corresponds to consider, in the representation space Hom(G,PGL(2, $\mathbb{C}$)), the faithful representations of G which are geometrically represented by noded quasiconformal deformations. This is a natural generalization for the deformation space of G, on which one considers the geometric representations given by quasiconformal homeomorphisms of G.

4.7 The case of Fuchsian groups: Bers–Abikoff's augmented Teichmüller space

Let us assume in here that G is a Fuchsian group of the first kind without parabolic elements, keeping the unit disc $\Delta_1 = \mathbb{D}$ invariant, and set $\Delta_2 = \widehat{\mathbb{C}} - \overline{\Delta_1}$. In this case, $\Omega(G) = \Delta_1 \cup \Delta_2$ and the Riemann orbifolds Δ_1/G and Δ_2/G are of signature $(g; n_1, \overset{r}{\ldots}, n_r)$, where $2 \le n_1 \le \cdots \le n_r < \infty$ and $n_1^{-1} + \cdots + n_r^{-1} < 2g + r - 2$. Associated to G we have the Teichmüllers spaces $T(\Delta_1, G)$, $T(\Delta_2, G)$ and $T(G) = T(\Omega(G), G)$. These are simply connected complex manifolds, the first two isomorphic and of complex dimensions $3g - 3 + r$, and the last one of complex dimension $6g - 6 + 2r$. In fact, there is an isomorphism

$$\theta : T(G) \to T(\Delta_1, G) \times T(\Delta_2, G) : [\mu] \mapsto ([\mu_1], [\mu_2]),$$

where μ_i is defined as μ on Δ_i and zero on its complement. Associated to the above three spaces are the corresponding partial closures given by the noded Teichmüller spaces $NT(\Delta_1, G)$, $NT(\Delta_2, G)$ and $NT(G)$. Each $NT(\Delta_j, G)$ can be identified with the augmented Teichmüller space of G as defined in [1–3,13]. The isomorphism θ extends continuously to an isomorphism

$$\theta : NT(G) \to NT(\Delta_1, G) \times NT(\Delta_2, G).$$

The Teichmüller space $T(\Delta_j, G)$ has associated the Weil–Petersson (WP) metric, which is Kähler, has negative sectional curvature, is not complete and for every pair of points there is a unique geodesic connecting them [5,53,57–59]. It is known that the WP metric completion of $T(\Delta_j, G)$ is the augmented Teichmüller space $NT(\Delta_j, G)$ [46] (but it is a non-locally compact space). Results on the geometry of geodesics on these spaces is given in [60]. The modular group $M(\Delta_j, G)$ acts a group of WP orientation-preserving isometries (these are all these isometries if $(g, r) \neq (1, 2)$ [60]). It was observed by Masur that $NT(\Delta_j, G)/M(\Delta_j, G)$ is the quotient WP metric completion $\overline{\mathcal{M}}(\Delta_j, G)$ (this being the Deligne–Mumford's compactification) of the moduli space $\mathcal{M}(\Delta_j, G)$ (the space of isomorphic classes of Riemann surfaces of genus g with r punctures).

In [32] it was proved that if M is a finite index subgroup of the modular group $M(\mathbb{D}, G)$, then the quotient spaces $NT(\Delta_1, G)/M$ and $NT(\Delta_2, G)/M$ are compact complex orbifolds. In particular, $NT(\Omega(G)/M$ is also a compact complex orbifold.

Remark 12 Now, following Remark 3 and the above, for every pair (Δ, G), where G is a finitely generated Kleinian group, and M being of finite index in $M(\Delta, G)$, it holds that $NT(\Delta, G)/M$ is a compact complex orbifold.

5 Uniformizations of (γ, n)-gonal pairs: proof of Theorem 1

Let us consider a (γ, n)-gonal pair (S, f) and let us denote by $B_f \subset R$ the locus of branch values of f. For each point in B_f we define its *branch order* as the minimum common multiple of the local degrees at all its preimages under f. Let us write $B_f = \{p_1, \ldots, p_r\}$ so that n_j is the branch order of p_j, where $2 \leq n_1 \leq n_2 \leq \cdots \leq n_r$. The *signature* of (S, f) is then defined by the tuple $(\gamma; n_1, \ldots, n_r)$. We assume, from now on, that (S, f) is of *hyperbolic type*, that is, $n_1^{-1} + \cdots + n_r^{-1} < 2\gamma + r - 2$ (in particular, if $\gamma = 0$, then $r \geq 3$). If S has genus g, then Riemann–Hurwitz formula asserts the equality

$$2(g - 1 - n(\gamma - 1)) = \sum_{j=1}^{r} \left(n - \# f^{-1}(p_j)\right).$$

Let us remark that topologically equivalent (γ, n)-gonal pairs have the same signature, but the converse is in general false.

For each point $q \in f^{-1}(p_j)$ set $m_q = n_j/d_q$, where d_q is the local degree of f at q, and let N_f be the set of of points $q \in f^{-1}(B_f)$ with $m_q > 1$. Let R^{orb} be the Riemann orbifold whose underlying Riemann surface structure is R, its cone point set is B_f and the cone order of each p_j is n_j. Similarly, let S^{orb} be the Riemann orbifold whose underlying Riemann surface structure is S and its cone points are given by the points $q \in N_f$ with cone order $m_q > 1$. As a consequence of the uniformization theorem: (i) there is a co-compact Fuchsian group G acting on the unit disc $\mathbb{D}$ and an orbifold isomorphism $\psi : R^{orb} \to \mathbb{D}/G$; that is, G has signature $(\gamma; n_1, \ldots, n_r)$, and (ii) there is an index n subgroup Γ and an orbifold isomorphism $\phi : S^{orb} \to \mathbb{D}/\Gamma$ so that $\psi f \phi^{-1} : \mathbb{D}/\Gamma \to \mathbb{D}/G$ is being induced by the inclusion $\Gamma < G$. The constructed pair of Fuchsian groups (Γ, G) is uniquely determined by the pair (S, f), up to conjugation by elements in $\mathrm{Aut}(\mathbb{D})$, the group of holomorphic automorphisms of $\mathbb{D}$; we say that (Γ, G) *uniformizes* (S, f). The following fact follows almost immediately from the definitions.

Lemma 3 *Two pairs of Fuchsian groups (Γ_1, G_1) and (Γ_2, G_2) uniformize topologically equivalent (γ, n)-gonal pairs (of hyperbolic type) if and only if there is an orientation-preserving homeomorphism $u : \mathbb{D} \to \mathbb{D}$ so that $uG_1u^{-1} = G_2$ and $u\Gamma_1u^{-1} = \Gamma_2$ (as the orbifolds are compact, we may assume u to be a quasiconformal homeomorphism). They correspond to twisted isomorphic pairs if $u \in \mathrm{Aut}(\mathbb{D})$ and to isomorphic ones if $G_1 = G_2$ and $u \in G_2$.*

Remark 13 Let us recall that a holomorphic automorphism of (S, f) is a holomorphic automorphism T of S so that $f = fT$. In this context, the group of holomorphic automorphisms of (S, f) is naturally identified with the quotient $N_G(\Gamma)/\Gamma$, where $N_G(\Gamma)$ is the normalizer of Γ in G.

As a consequence of Lemma 3 one obtains that $\mathcal{H}_0(S, f)$ can be identified with the Teichmüller space $T(\mathbb{D}, G)$ of the orbifold $\mathbb{D}/G$, which is a simply-connected complex manifold of dimension $3\gamma + r - 3$ [38,40,49]. As every point in $\mathcal{H}(S, f)$ is uniformized by a quasiconformal deformation of (Γ, G), is a consequence of the measurable Riemann mapping theorem [6,48] that the Hurwitz space $\mathcal{H}(S, f)$ is connected and it can be identified with $T(\mathbb{D}, G)/M(\Gamma, G)$, where $M(\Gamma, G)$ is a subgroup of the modular group $M(\mathbb{D}, G)$ (the group of holomorphic automorphisms of $T(\mathbb{D}, G)$) induced by the isotopic classes of those quasiconformal self-homeomorphisms of $\mathbb{D}$ normalizing both G and also Γ. The above, in particular, asserts that $\mathcal{H}(S, f)$ is a connected complex orbifold of dimension $3\gamma + r - 3$ whose orbifold fundamental group is isomorphic to $M(\Gamma, G)$.

Remark 14 As the subgroup $M(\Gamma, G)$ has finite index in the modular group $M(\mathbb{D}, G)$, there is a finite degree branched covering $\mathcal{H}(S, f) \to \mathcal{M}(\mathbb{D}, G)$, where $\mathcal{M}(\mathbb{D}, G)$ is the moduli space of the orbifold $\mathbb{D}/G$ (this being the moduli space of an r-punctured Riemann surface of genus γ). For instance, if Γ is a characteristic subgroup of G, then $M(\Gamma, G) = M(\mathbb{D}, G)$, so $\mathcal{H}(S, f) = \mathcal{M}(\mathbb{D}, G)$.

6 Uniformizations of stable (γ, n)-gonal pairs: proof of Theorem 2

In this section we proceed to construct uniformizations of an stable (γ, n)-gonal pair using noded Fuchsian groups. These uniformization permits to provide a compactification $\overline{\mathcal{H}}(S, f)$, for (S, f) a (γ, n)-gonal pair of hyperbolic type, this being a complex orbifold containing $\mathcal{H}(S, f)$ as an open dense suborbifold.

Let us recall that each stable (γ, n)-gonal pair (S^*, f^*) is obtained by considering a suitable (γ, n)-gonal pair $(S, f : S \to R)$ of hyperbolic type and a suitable collection $\mathcal{F} = \{\gamma_1, \ldots, \gamma_s\}$ of pairwise disjoint simple loops in $R - B_f$, where B_f is the set of branch values of f. If r is the cardinality of B_f, then for $\gamma = 0$ we also assume that $r \geq 4$ (otherwise, there is no possible degeneration to make). The collection $\mathcal{F}$ satisfies that the Euler characteristic of each connected component of $R - (B_f \cup \gamma_1 \cup \cdots \cup \gamma_s)$ is negative and none of the components of $R - (\gamma_1 \cup \cdots \cup \gamma_s)$ is a disc with only two branch values, both with branch order equal to 2. By the pinching process of the loops in $\mathcal{F}$ and those in $f^{-1}(\mathcal{F})$ (as indicated in the introduction), we obtain a topological stable (γ, n)-gonal pair homeomorphic to (S^*, f^*). To obtain (S^*, f^*) we need to provide to each component of the nodes of a suitable analytically finite Riemann surface.

Let us consider a pair (Γ, G) of Fuchsian groups uniformizing (S, f), that is, there are orbifold isomorphisms $\phi : S^{orb} \to \mathbb{D}/\Gamma$ and $\psi : R^{orb} \to \mathbb{D}/G$ so that $\pi := \psi f \phi^{-1}$ is branched cover induced by the inclusion of Γ in G. Recall that the pair (Γ, G) is unique up to conjugation by elements of $\mathrm{Aut}(\mathbb{D})$. The collection of simple loops $\psi(\mathcal{F}) \subset \mathbb{D}/G$ lifts under π to a collection $\mathcal{G}$ of simple arcs in $\mathbb{D}$ which is pinchable for G, so also for the subgroup Γ (see Remark 8). Set $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$ and let us consider three different points $a, b, c \in \overline{\mathbb{D}}$. Now, as a consequence of Propositions 3 and 4, there is a $\mu \in L^{\infty}_{noded}(\mathbb{D}, G)$ with limit set being the closure of the collection $\mathcal{G}$. If $w_\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a noded quasiconformal deformation for G with the complex dilation μ and fixing the points a, b, c, then Theorem 7 asserts the existence of a unique Kleinian group $\theta(G)$ and a unique isomorphism of groups $\theta : G \to \theta(G)$ such that $w_\mu \circ \gamma = \theta(\gamma) \circ w_\mu$. Moreover, the region of discontinuity for the action of $\theta(G)$ on the Riemann sphere is $w_\mu(\Omega(\mu) \cap \Omega(G)) = w_\mu(\Omega(\mu) \cap \mathbb{D}) \cup w_\mu(\overline{\mathbb{D}})$. Observe that $w_\mu(\overline{\mathbb{D}})$ is a quasidisc. Corollary 5 asserts that $\theta(G)$ (and so $\theta(\Gamma)$) is a noded Fuchsian group. By Proposition 3, the pair $(\theta(\Gamma), \theta(G))$, when restricted to $w_\mu(\Omega(\mu) \cap \mathbb{D})$, provides a stable (γ, n)-gonal pair modelled by (S^*, f^*). Also, the same pair, when restricted to $w_\mu(\overline{\mathbb{D}})$ provides the (γ, n)-gonal pair (S, f). Up to post-composition by a suitable quasiconformal homeomorphisms, we may assume that the stable (γ, n)-gonal pair we obtain is the original one, i.e., (S^*, f^*); we say that the pair $(\theta(\Gamma), \theta(G))$ *uniformizes* it. For each $t \in \mathbb{D}$, we may consider $t\mu \in L^{\infty}_{1}(\mathbb{D}, G)$ and the quasiconformal homeomorphism $w_{t\mu} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with complex dilation $t\mu$ and fixing the points a, b, c. If $G_t = w_{t\mu} G w_{t\mu}^{-1}$ and $\Gamma_t = w_{t\mu} \Gamma w_{t\mu}^{-1}$, then (Γ_t, G_t) provides a continuous family of (γ, n)-gonal pairs (S_t, f_t) converging to (S^*, f^*).

The above construction asserts that the Hurwitz space $\overline{\mathcal{H}}_0(S, f)$, parametrizing isomorphic classes of stable (γ, n)-gonal pairs, modelled by degeneration of a the (γ, n)-gonal pair (S, f), can be identified with the noded Teichmüller space $NT(\mathbb{D}, G)$. Similarly, the Hurwitz space $\overline{\mathcal{H}}(S, f)$, parametrizing twisted isomorphic classes of stable (γ, n)-gonal pairs, modelled by degeneration of a the (γ, n)-gonal pair (S, f), can be identified with the quotient space $NT(\mathbb{D}, G)/M(\Gamma, G)$. As $M(\Gamma, G)$ has finite index in the modular group $M(\mathbb{D}, G)$, it follows from the results of [32] that $NT(\mathbb{D}, G)/M(\Gamma, G)$ carries a structure of a complex orbifold, being a finite branched cover of the Deligne–Mumford's compactification space of $\mathbb{D}/G$.

Remark 15 The forgetful map (of finite degree) $\mathcal{H}(S, f) \to \mathcal{M}_g : [(S', f')] \mapsto [S']$ extends to the corresponding forgetful map (also of finite degree) $\overline{\mathcal{H}}(S, f) \to \widehat{\mathcal{M}}_g : [(S', f')] \mapsto [S']$.

Acknowledgements The author is very grateful to the referee, in special for pointing out some references which improved the content and clarity in the presentation of this paper.

Compliance with ethical standards

Conflict of interest The author declares that he has no conflicts of interest.

References

1. Abikoff, W.: On boundaries of Teichmüller spaces and on Kleinian groups, III. Acta Math. **134**, 211–237 (1975)
2. Abikoff, W.: Degenerating families of Riemann surfaces. Ann. Math. (2) **105**, 29–44 (1977)
3. Abikoff, W.: Augmented Teichmüller spaces. Bull. Am. Math. Soc. **82**(2), 333–334 (1976)
4. Ahlfors, L.V.: Finitely generated Kleinian groups. Am. J. Math. **86**, 413–429 (1964). correction *ibid***87** (1965), 759
5. Ahlfors, L.V.: Some remarks on Teichmüller's space of Riemann surfaces. Ann. Math. (2) **74**, 171–191 (1961)
6. Ahlfors, L., Bers, L.: Riemann's mapping theorem for variable metrics. Ann. Math. **72**, 385–404 (1960)
7. Arfeux, M.: Berkovich spaces and Deligne–Mumford compactification. arXiv:1506.02552
8. Bartolini, G., Costa, A.F., Izquierdo, M.: On automorphisms groups of cyclic p-gonal Riemann surfaces. J. Symb. Comput. **57**, 61–69 (2013)
9. Beardon, A.E.: The Geometry of Discrete Groups. Graduate Texts in Mathematics, vol. 91. Springer, New York (1983)
10. Belyi, G.V.: On Galois extensions of a maximal cyclotomic field. Math. USSR Izv. **14**, 247–265 (1980)
11. Bers, L.: Simultaneous uniformization. Bull. Am. Math. Soc. **66**(2), 94–97 (1960)
12. Bers, L.: Spaces of Kleinian groups. In: Several Complex Variables, Maryland. Lecture Notes in Mathematics, vol 155, pp. 9–34. Springer, Berlin (1970)
13. Bers, L.: On boundaries of Teichmüller spaces and on Kleinian groups. I. Ann. Math. Second Ser. **91**, 570–600 (1970)
14. Bers, L.: On spaces of Riemann surfaces with nodes. Bull. Am. Math. Soc. **80**, 1219–1222 (1974)
15. Bujalance, E., Costa, A.F.: Automorphism groups of cyclic p-gonal pseudo-real Riemann surfaces. J. Algebra **440**, 531–544 (2015)
16. Clebsch, A.: Zur Theorie der Riemann'schen Fläche. Math. Ann. **6**(2), 216–230 (1873)
17. Coelho, J., Sercio, F.: On the gonality of stable curves. arXiv:1507.07494v2.pdf
18. Costa, A.F.: One example of 1-dimensional Hurwitz space. In: López, J.L., Coronado, L.M.P. (eds.) Mathematical Contributions: Volume in Honor of Professor Joaquín Arregui Fernández (Spanish), pp. 127–134. Homenajes de la Universidad Complutense, Editorial Complutense, Madrid (2000)
19. Costa, A.F., González-Aguilera, V.: Limits of equisymmetric 1-complex dimensional families of Riemann surfaces. Math. Scand. **121**(1), 26–48 (2017)
20. Costa, A.F., Izquierdo, M., Parlier, H.: Connecting p-gonal loci in the compactification of moduli space. Rev. Mat. Complut. **28**(2), 469–486 (2015)
21. Costa, A.F., Izquierdo, M., Riera, G.: One-dimensional Hurwitz spaces, modular curves, and real forms of Belyi meromorphic functions. Int. J. Math. Math. Sci. **2008**, 1–18 (2008)
22. Deligne, P., Mumford, D.: The irreducibility of the space of curves of given genus. Inst. Hautes Études Sci. Publ. Math. **36**, 75–109 (1969)
23. Diaz, R., González-Aguilera, V.: Limit points of branch locus of M_g. Adv. Geom. (to appear)

24. Earle, C.J.: Teichmüller spaces and complex manifolds. Warwick 1992. Teichmüller Theory Moduli Spaces **10**, 5–33 (2010)
25. Eisenbud, D., Elkies, N., Harris, J., Speiser, R.: On the Hurwitz scheme and its monodromy. Compos. Math. **77**(1), 95–117 (1981)
26. Fried, M.: Fields of definition of function fields and Hurwitz families—groups as Galois groups. Commun. Algebra **5**(1), 17–82 (1977)
27. Gromadzki, G., Weaver, A., Wootton, A.: On gonality of Riemann surfaces. Geom. Dedicata **149**, 1–14 (2010)
28. Harris, J., Mumford, D.: On the Kodaira dimension of the moduli space of curves. Invent. Math. **67**, 23–86 (1982)
29. Hidalgo, R.A.: The noded Schottky Space. Proc. Lond. Math. Soc. **73**, 385–403 (1996)
30. Hidalgo, R.A.: Noded Fuchsian groups I. Complex Var. Theory Appl. **36**, 45–66 (1998)
31. Hidalgo, R.A., Vasil'ev, A.: Noded Teichmüller spaces. J. Anal. Math. **99**(1), 89–107 (2006)
32. Hinich, V., Vaintrob, A.: Augmented Teichmüller spaces and orbifolds. Sel. Math. (N.S.) **16**(3), 533–629 (2010)
33. Hubbard, J.H., Koch, S.: An analytic construction of the Deligne–Mumford compactification of the moduli space of curves. J. Differ. Geom. **98**, 261–313 (2014)
34. Hurwitz, A.: Über Riemann'sche Flächen, mit gegebenen Verzweigungspunkten. Math. Ann. **39**(1), 1–60 (1891)
35. Keen, L., Maskit, B., Series, C.: Geometric finiteness and uniqueness for Kleinian groups with circle packing limit sets. J. Reine Angew. Math. **436**, 209–219 (1993)
36. Koebe, P.: Über die Uniformisierung der Algebraischen Kurven II. Math. Ann. **69**, 1–81 (1910)
37. Kra, I.: On spaces of Kleinian groups. Comment. Math. Helv. **47**, 53–69 (1972)
38. Kra, I., Maskit, B.: The deformation space of a Kleinian group. Am. J. Math. **103**, 1065–1102 (1981)
39. Kra, I., Maskit, B.: Pinching two component Kleinian groups. In: Cazacu, C.A., Lehto, O.E., Rassias, T.M. (eds.) Analysis and Topology, pp. 425–465. World Scientific Press, London (1998)
40. Letho, O.: Univalent Functions and Teichmüller Spaces. Graduate Texts in Mathematics. Springer, Berlin (1986)
41. Maskit, B.: Kleinian Groups. Grundlehren der Mathematischen Wissenschaften. Springer, Berlin (1988)
42. Maskit, B.: On boundaries of Teichmüller spaces II. Ann. Math. **91**, 607–639 (1970)
43. Maskit, B.: On a class of Kleinian groups. Ann. Acad. Sci. Fenn. Ser. A I Math. **442** (1969). http://www.acadsci.fi/mathematica/1969/no442pp01-08.pdf
44. Maskit, B.: Parabolic elements in Kleinian groups. Ann. Math. **117**, 659–668 (1983)
45. Maskit, B.: Self-maps of Kleinian groups. Am. J. Math. **93**, 840–856 (1971)
46. Masur, H.: Extension of the Weil–Petersson metric to the boundary of Teichmüller space. Duke Math. J. **43**(3), 623–635 (1976)
47. Moore, R.L.: Concerning upper semi-continuous collection of continua. Trans. AMS **27**, 412–428 (1925)
48. Morrey Jr., C.B.: On the solutions of quasi-linear elliptic partial differential equations. Trans. Am. Math. Soc. **43**(1), 126–166 (1938)
49. Nag, S.: The Complex Analytic Theory of Teichmüller Spaces. Canadian Mathematical Society, Series of Monographs and Advances Texts. Wiley, New York (1988)
50. Natanzon, S.M.: Uniformization of spaces of meromorphic functions. Sov. Math. Dokl. **33**, 487–490 (1986)
51. Natanzon, S.M.: Fuchsian groups and uniformization of Hurwitz spaces. Disertaciones del Seminario de Matemáticas Fundamentales, vol. 19, Universidad Nacional de Educación a Distancia. https://www.researchgate.net/publication/28272129 (1999)
52. Ohshika, K.: Geometrically finite Kleinian groups and parabolic elements. Proc. Edinb. Math. Soc. (2) **41**(1), 141–159 (1998)
53. Royden, H.L.: Intrinsic metrics on Teichmüller space. In: Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), vol. 2, pp. 217–221. Canadian Mathematical Congress, Montreal, Quebec (1975)
54. Severi, F.: Vorlesungen über algebraische Geometrie. Teubner-Verlag, Leipzig (1921)
55. Sullivan, D.: On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions, Riemann surfaces and related topics. Ann. Math. Stud. **97**, 465–496 (1981)

56. Thurston, W.P.: On the geometry and dynamics of diffeomorphisms of surfaces. Am. Math. Soc. Bull. New Ser. **19**(2), 417–431 (1988)
57. Tromba, A.J.: On a natural algebraic affine connection on the space of almost complex structures and the curvature of Teichmüller space with respect to its Weil–Petersson metric. Manuscripta Math. **56**(4), 475–497 (1986)
58. Wolpert, S.A.: Chern forms and the Riemann tensor for the moduli space of curves. Invent. Math. **85**(1), 119–145 (1986)
59. Wolpert, S.A.: Geodesic length functions and the Nielsen problem. J. Differ. Geom. **25**(2), 275–296 (1987)
60. Wolpert, S.A.: Geometry of the Weil–Petersson completion of the Teichmüller space. Surv. Differ. Geom. **8**, 357–393 (2003)